Order Number 9308319

Microeconomic simulation output analysis using a two-way random effects metamodel

Gentner, Frederick John, Ph.D.

The University of Michigan, 1992

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

MICROECONOMIC SIMULATION OUTPUT ANALYSIS USING A TWO-WAY RANDOM EFFECTS METAMODEL

by

Frederick John Gentner

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Business Administration)
in The University of Michigan
1992

Doctoral Committee:

Associate Professor Richard W. Andrews, Chairman
Associate Professor William C. Birdsall
Professor Thomas J. Schriber
Professor William J. Wrobleski

To Tarb

ACKNOWLEDGEMENTS

I wish to thank Professors Bill Birdsall, Tom Schriber and Bill Wrobleski, and especially my committee chairman Professor Andy Andrews, for their assistance in completing this work; their interest, encouragement and comments are deeply appreciated. I also wish to thank Professor Peter Lenk for his help in explaining some of the technical aspects of asymptotic expansions.

TABLE OF CONTENTS

LIST OF TABLES

Table

LIST OF FIGURES

Figure

LIST OF APPENDICES

Appendix

CHAPTER 1

INTRODUCTION

Microsimulation models of economic systems have become widely used in recent years. Like any simulation model, they are most powerful when used within the context of a statistically designed experiment. Unfortunately, not much attention has been given to the experimental design of microsimulation models, particularly to the use of replicated observations on the models. The structure of the two-way random effects model naturally lends itself to use as a metamodel for the analysis of the output from microsimulation experiments. The use of Bayesian analysis permits the incorporation of the model user's experience and knowledge into the analysis by use of prior distributions on model parameters.

Chapter 1 contains an introduction and overview of this work. An example microsimulation model is presented in Chapter 2. The two-way random effects model is presented as a useful metamodel for the microsimulation experiment in Chapter 3, which includes a comparison of the frequentist theory and Bayesian theory approaches to the analysis of this model. The Bayesian methodology is developed in Chapters 4 and 5. Demonstrations of the Bayesian analysis methodology are presented in Chapter 6. A summary and consideration of further research possibilities are presented in Chapter 7.

In Section 1.1 the general nature of microsimulation models and some experimental design issues are discussed. In Section 1.2 the two-way random effects model is proposed as an appropriate metamodel for microsimulation experiments. Section 1.3 discusses the use of the mean response of an as yet unobserved replication of the microsimulation model as the system performance measure. In Section 1.4 the relative merits of Bayesian theory and frequentist theory approaches to inference are compared. And in Section 1.5 an overview is presented, describing the objective, methodology, conclusions and impact of this work.

1.1 Microsimulation Models

Microeconomic simulation models are simulation models of microeconomic decision units; they are also referred to as microsimulation models, used throughout the remainder of this work, as well as microanalytic models or microdata simulation models. They are computer-implemented, stochastic models of the behavior of heterogeneous economic decision units in an economic environment over time. The decision units have descriptive characteristics which are stochastically updated in response to the economic environment; the state of the environment is represented by model parameters, referred to as operating characteristics. Commonly used decision units are individuals, households, business firms, industries, and government units. Typically, the collection of individual unit characteristics are aggregated, in any particular time period, to describe the overall state of the economy.

Microsimulation models can be described as Monte Carlo sampling distribution models. They are different from the simulation models of dynamic queueing systems which have homogeneous traffic units simultaneously competing for scarce resources. Also, they are different in that simulation queueing models are highly dependent on event scheduling. The microsimulation models contain traffic units that are heterogeneous microeconomic decision units; subsequently, the term *decision unit* will be used to refer to a traffic unit in a microsimulation model. The decision units travel recursively through the model, making a pass through the model for each time period, without interacting or competing with other decision units. During each pass, each decision unit's descriptive characteristics are stochastically modified in response to the state of the economy. Microsimulation models are run for a specific number of time periods, so they are treated as *terminating condition* simulation models, not *steady state* models. As such, multiple observations on system performance measures are obtained by the method of replications, requiring that each replication of the model begins with an identical initial state but uses a different set of random numbers. The identical initial state requires that the same set of operating characteristics and the same set of decision units, each with the same initial set of descriptive characteristics, are used.

Microsimulation models are used widely in government offices, universities and private research institutions, and private contractors; the models are used in the United States, Canada, and several European nations. In the United States government, microsimulation models are used by various departments and agencies, such as the Congressional Budget Office, the Joint Tax Committee, Office of Tax Analysis of the Treasury Department, the Department of Health and Human Services, the Department of Agriculture, and the Office of Management and Budget. Indeed, Betson (1990, p. 425) stated that the majority of budget estimates are produced by microsimulation techniques. For examples of uses of microsimulation models see: Orcutt, Caldwell, and Wertheimer (1976);

Haveman and Hollenbeck (1980); Feldstein (1983); Nakamura and Nakamura (1985b); Bennett and Bergmann (1986); Kraemer and King (1986); Orcutt, Merz, and Quinke (1986); and Brunner and Petersen (1990).

In Chapter 2, a microsimulation model of the labor-force participation of married women is described. This model is based upon one of the models given in Nakamura and Nakamura (1985a); it is used as the example microsimulation model throughout this work. In this model, the decision unit is a married woman, subsequently referred to as a wife, and the dependent variable is the wife's annual earnings. The sample output from an experiment with this model is used as one of the example data sets in Chapter 6, to demonstrate the Bayesian methodology developed in Chapters 4 and 5.

From the earliest examples, users have recognized that running a microsimulation model is a statistical experiment and that the model results are inherently variable. However, the reported uses of various models typically describe the simulation results in terms of point estimates, rarely reporting confidence bands or other interval estimates. It appears that the efforts of the microsimulation researchers have mainly gone into building the models and estimating the parameters, interesting issues when wearing the economist's hat. Little effort has been given to issues of experimental design, interesting issues when wearing the statistician's hat. Early on, Naylor, Burdick and Sasser (1967) commented that economists have virtually ignored experimental design considerations of simulation modeling, perhaps due to having had only limited opportunities to perform experiments with economic systems before the advent of simulation modeling. Over the years, things apparently have not improved, prompting Kenneth Arrow to comment on the lack of commonplace statistical inference practices at a 1978 Washington, D. C., conference for model builders, policy makers, and the academic community:

> Unfortunately, as far as I can see, in all uses of models for policy purposes (including those at this conference) there is no confidence or error band. (Arrow, 1980, p. 260)

In a report of the National Research Council (1991), the Panel to Evaluate Microsimulation Models for Social Welfare Programs strongly recommended that information about the levels and sources of uncertainty in policy analysis work be routinely included in the reports provided to model users.

The variation in microsimulation model output may be classified as arising from three sources: (1) Monte Carlo variation, (2) decision unit sample variation, and (3) modeling variation. Orcutt, Greenberger, Korbel and Rivlin (1961) discussed similar sources of variation, without using these particular labels.

(1) Monte Carlo variation refers to the variation arising from the use of random numbers in the model. Monte Carlo variation occurs in a single replication of a model since random numbers are used to decide whether events will or will not occur, such as birth, death, marriage, divorce, or entry into the labor force; random numbers may also be used to determine the stochastic deviation from the expected value of the level of a decision unit's characteristic, such as wage rate or number of hours worked. Monte Carlo variation also arises from performing replications of a model since the output in each replication depends on the particular set of realized values of the random numbers.

(2) Decision unit sample variation refers to the variation arising from the use of a subset of the population of interest as decision units in the microsimulation model. In models with individuals or households as the decision units, it is common to have a population with size in the millions represented by a sample with size in the thousands. The use of a sample rather than the population reduces the time needed to run each replication of the model and reduces the data collection requirements. Values are needed for all descriptive characteristic of each decision unit. This information usually must be accumulated from a number of sources.

(3) Modeling variation is intended to encompass all other possible sources of variation including, but not limited to, variation due to operating characteristics estimation, imputation of values for decision unit characteristics, and model specification error.

Even if modeling variation can be eliminated, by assuming that a perfect model with known parameters is specified, the first two sources of variation would remain. And further, even if sampling variation is eliminated by using the entire population of interest in the model, Monte Carlo variation would still remain. Monte Carlo variation is the heart of simulation modeling experiments; the nature of stochastic simulation models is to exploit and explore the empirical distributions of output variables generated by Monte Carlo variation. Unfortunately, it is the least discussed source of variation in the economics literature. Arrow (1980, p. 260) emphasized the need to deal with Monte Carlo variation:

> What is needed is replication, repeated observations within a time series or a cross-section context ... . So it has to be understood that even direct observation should be tested by repeated observations, at several points in time or for several individuals. (p. 260)

When the different sources of variation are directly addressed in the economics literature, there seems to be confusion as to how to deal appropriately with them. For example, Orcutt, et al. (1961) suggested using replications, which contribute to Monte Carlo variation, to address the issue of decision unit sample variation:

> (I)t is still necessary ... to approximate the real system of millions of units with a reduced system containing thousands of units ... . One solution would be to do quite a number of runs with the same initial population and the same operating characteristics and get a distribution of final results, from which could be estimated the expected numbers of units of given characteristics and the variances of these numbers. (pp. 32-33)

As will be explained in this work, a well designed statistical experiment allows the model user to identify, and thereby properly evaluate, the different sources of variation.

A reason for using microsimulation models of large socioeconomic systems is the ability to perform various types of experiments on the computerized model that would be impractical or impossible to perform on the actual systems. These experiments include (1) projections of the state of the economy into the future, (2) investigation of the effects of alternative economic policies on the state of the economy, (3) sensitivity analysis, with respect to the model specification or operating characteristics, and (4) generation of decision unit histories for the investigation of the impact of policy decisions on the behavior of individual units; see Orcutt, et al. (1976). The purpose for using a model influences the selection of the response variable, the design of the experiment, and the nature of the statistical inference procedure employed.

1.2 Two-Way Random Effects Model as a Simulation Metamodel

A simulation metamodel is a mathematical model, usually less complicated than the simulation model itself, which is used to analyze the simulation output. The metamodel is selected to reflect the nature of the output data and the objectives of the experiment. The two-way random effects model is often an appropriate metamodel for the output from a microsimulation model.

The output from a microsimulation model experiment is a matrix of observed values for the dependent variable. Each column vector in the matrix corresponds to the outcomes from a single replication, with one element for each decision unit in the sample. Each row vector corresponds to the outcomes from a single decision unit, with one element for each replication. This matrix structure of data values is matched by the structure of the two-way random effects model.

The two-way random effects model determines the value of the dependent variable as a linear function of a constant, two random effects, and a random error term. The model is

$$Y_{ij} = \psi + R_i + C_j + E_{ij}\,, \tag{1.1}$$

where Y_{ij} is the measurement on the characteristic of interest for the i^{th} decision unit in the j^{th} replication, and ψ is the overall mean. The following distribution assumptions are made for the random variables on the right side:

the row effect, $R_i \sim \text{Normal}\left(0, \sigma_R^2\right)$;

the column effect, $C_j \sim \text{Normal}\left(0, \sigma_C^2\right)$; and,

the error term, $E_{ij} \sim \text{Normal}\left(0, \sigma_E^2\right)$.

It is further assumed that the row effects, column effects and error terms are statistically independent over all {i, j}. The structure of this model permits the identification of decision unit sample variation with the row effects, Monte Carlo variation with the column effects, and modeling variation with the error term. Thus, the use of the two-way random effects model permits the different sources of variation to be identified, separated, and investigated.

1.3 Inference About the Mean of a New Replication

The objectives of a simulation experiment determine which system performance measure is appropriate in any particular application. A parameter, or a function of the parameters, of the metamodel is selected as the appropriate system performance measure. The sample observations from the replications of the model are used to make an inference about the system performance measure.

In this work, it is assumed that the model user is interested in the mean of a randomly occurring, as yet unobserved, replication of the model. Define the variable X_j as the mean of the j^{th} replication, where the expectation is taken over the row/decision unit dimension:

$$X_j = E_i\left[Y_{ij}\right]$$

$$= E_i\left[\psi + R_i + C_j + E_{ij}\right]$$

$$= \psi + C_j \,. \qquad (1.2)$$

Incorporating this definition into the model given in Equation (1.1), the two-way random effects model may be expressed:

$$Y_{ij} = R_i + X_j + E_{ij} \,,$$

where the random column effect $X_j \sim \text{Normal}\left(\psi, \sigma_C^2\right)$. The experiment is performed over I decision units and J replications, resulting in an I×J matrix of observed values. Without loss of generality, any unobserved, randomly selected replication may be referred to as the $(J+1)^{th}$ replication. So, it is assumed that the model user is interested in the mean of the $(J+1)^{th}$ replication, X_{J+1}. Using the Nakamura model as an example, with the annual earnings of a wife as the dependent variable, it is assumed that the model user is interested in the mean of annual earnings of all wives for the $(J+1)^{th}$ replication.

The reason for selecting the mean of the $(J+1)^{th}$ replication, X_{J+1}, rather then the overall mean, ψ, as the system performance measure is based upon the nature of the economic systems upon which microsimulation models are based. Microsimulation models are models of economic systems in the real world. Conceptually an economic system can be considered a stochastic process with, at any point in time, a random distribution of possible states that may occur in the future. An economic system has only a single realized time path, not multiple realized observations. The overall mean, ψ, describes the average annual earnings of all wives over all possible replications of the model; conceptually, there are an infinite number of possible replications. The mean of the $(J+1)^{th}$ replication, X_{J+1}, describes the average annual earnings of all wives for a single replication, just as an economic system has but a single realization.

Interval estimates for the mean of the $(J+1)^{th}$ replication, X_{J+1}, are developed in this work. The frequentist theory confidence intervals are based on the sampling distribution of the overall mean; this approach is developed in Section 3.3. The Bayesian theory credible sets are based on the posterior distribution of X_{J+1}, this approach is outlined in Section 3.4 and developed in detail in Chapter 4. The posterior distribution can be used as the basis for any type of Bayesian inference (Berger, 1985). Specific intervals can be constructed from the posterior distribution ; in particular, highest posterior density (HPD) credible sets are constructed, analogous to the frequentist's confidence intervals.

1.4 Bayesian and Frequentist Analysis

Difficulties exist with the implementation of either the frequentist or Bayesian inference approach. Some problems which arise when using the frequentist approach are described first, and then some problems encountered using the Bayesian approach are described.

A major problem with the frequentist analysis of the two-way random effects model is the possibility of obtaining negative estimates for the row effect variance, column effect variance, or both, when using the standard method-of-moments estimators. These estimators are found by equating the expected mean squares values from the analysis of variance table with the corresponding variance functions. The estimators are uniformly minimum variance and unbiased for a balanced design when using normal distributions (Searle, 1971). When negative estimates occur, "several courses of action exist, few of them satisfactory," (Searle, 1971, p. 407). Among the courses of action specifically mentioned, Searle included using alternative estimation methods such as maximum likelihood analysis or Bayesian analysis. Maximum likelihood estimators for the one-way random effects model have been found, but these estimators are biased. Due to the complexity of the likelihood function for the two-way random effects model, Equation (1.1), there are no closed form solutions for the maximum likelihood estimators (Szatrowski and Miller, 1980, pp.814-815).

In Bayesian analysis, each parameter is restricted to its probability space; for a variance, this is the non-negative portion of the number line. Estimates of a variance are based upon its posterior distribution which is only defined for non-negative values. Bayesian analysis for the two-way random effects model has been addressed generally in Box and Tiao (1973) and Broemeling (1985). In those works, interest is focused primarily on the variance parameters, with the mean being considered a nuisance parameter; analytic results for the marginal posterior distributions of the variances are not available due to the intractability of the integrals encountered.

Among the criticisms of Bayesian analysis, two are specifically mentioned here: (1) solicitation of prior distributions; and (2) intractability of integrals. Bayesian analysis permits the incorporation of the model user's experience, knowledge and common sense into the analysis by the use of prior distributions on model parameters. The determination of the posterior distribution by combining the prior distribution and likelihood function through Bayes' theorem is generally not feasible, except when using conjugate priors. Even when using conjugate priors, the solicitation from the model user of the parameter values for these prior distributions may be a problem. This work assumes that the model user can specify the parameter values for the conjugate prior distributions used in the analysis.

The problem of intractable integrals may arise in Bayesian analysis, especially when using multiple parameter models such as the two-way random effects model. The problem occurs because of the need to integrate over nuisance parameters in a joint posterior distribution to obtain the marginal posterior distribution of the parameter of interest. Methods of dealing with the intractable integrations include numerical integration, Monte Carlo integration, analytic approximations, and Gibbs sampling. A particular type of analytic approximation, proposed by Tierney and Kadane (1986), is based on LaPlace's method for integral approximation, using a Taylor series expansion about the mode of the distribution of the nuisance parameters.

In this work, methods for the approximation of intractable integrals encountered in the derivation of the posterior distribution of the mean response of the $(J+1)^{th}$ replication from a two-way random effects model are developed; these methods are referred to as *analytic-numeric approximations*. The analytic portion of this method derives a function of the integrand which, when evaluated at the mode, approximates the value of the integral; however, the mode cannot be determined analytically. The numeric portion of this method locates the mode of the integrand, and evaluates the approximation function, numerically. While the two-way random effects model does not satisfy the regularity conditions for guaranteeing the validity of LaPlace's method in all applications (Kass, Tierney and Kadane, 1990), the analytic-numeric approximations do work in the examples presented later of the Nakamura model.

1.5 Overview

The objective of this work is the demonstration of Bayesian procedures for the analysis of output from microsimulation models. In order to accomplish this, a Bayesian estimation methodology is developed for the mean of the dependent variable for a randomly selected column of the two-way random effects model. Using the mean response of the $(J+1)^{th}$ replication of the microsimulation model as the system performance measure focuses the attention of the model user on an appropriate measure matching the behavior of the real system being studied.

Using conjugate prior distributions for the model parameters, analytic Bayesian solutions are used as far as possible. The model likelihood function is combined with the joint prior distribution by Bayes' theorem to obtain the joint posterior distribution of the mean of the $(J+1)^{th}$ column and the four model parameters, $\left\{\psi, \sigma_R^2, \sigma_C^2, \sigma_E^2\right\}$. Integration over the mean parameter is performed analytically. The integrations over the three variance parameters are not tractable. An analytic-numeric approximation for these integrations is developed following the LaPlace method for integral approximation, where the modes and solution of the approximation function are performed numerically.

A microsimulation model of the labor force participation of wives, with annual earnings as the dependent variable, is used as an example throughout. For this example model, the mean annual earnings of the $(J+1)^{th}$ replication is the system performance measure. Output from the example model is used to demonstrate the analytic-numeric approximation method for finding selected Bayesian HPD credible sets for the system performance measure, using non-informative and informative prior distributions for the model parameters. Corresponding frequentist confidence intervals are presented for comparison. The comparative analyses are performed for the different situations which can arise in practice, when estimates for the row variance or column variance or both have negative values. The comparisons demonstrate the different results that can occur when using the same experimental results with the different philosophical approaches to inference. It is shown that analogous interval estimates have different midpoints and different widths, reflecting the different estimates of means and standard deviations resulting from the use of Bayesian or sampling theory methods.

The use of microsimulation models can be enhanced by employing the two-way random effects model as a simulation metamodel, and the mean of the $(J+1)^{th}$ replication as the system performance measure about which inference is made. Employing Bayesian analysis permits the model user a systematic way to incorporate the user's prior knowledge about the behavior of the system being investigated. The analytic-numeric approximation method developed and demonstrated in this work provides a computer based method for accomplishing the Bayesian analysis of simulation model output which may be useful in many situations.

CHAPTER 2
THE SIMULATION MODEL AND EXPERIMENT

This Chapter presents the simulation model and experiment used as the primary example throughout this work. The microsimulation model is described in Section 2.1. The decision unit sample is discussed in Section 2.2. The computer program which implements this model is discussed in Section 2.3. And, the results of the simulation experiment are discussed in Section 2.4.

2.1 The Nakamura Microsimulation Model

The microsimulation model used to demonstrate the application of a two-way random effects Bayesian analysis is from Nakamura and Nakamura (1985a). That article compared three models of the labor force participation of married women, each model incorporating different amounts of past information. Of the three, the *Difference model* is used as the example microsimulation model in this work, and referred to hereafter as the *Nakamura model.*

The Nakamura model is a model of the labor force participation of married women with a time period of one calendar year. The dependent variable for each wife is her annual earnings. Annual earnings are determined in a three step stochastic process: the first step, called the *Probit Index* step, determines whether or not the wife is working during the year; the second step, called the *Wage Rate* step, determines the hourly wage rate received during the year; and the third step, called the *Hours Worked* step, determines the number of hours worked during the year. The explanatory variables in each of the three steps include personal characteristics, family characteristics, and macroeconomic characteristics. Table 2.1 contains a complete list of the model explanatory variables and their classification. The table designation as an individual or a family characteristic is determined by the type of record where the information is located in the Panel Study of Income Dynamics (PSID). This data set is collected and published by the Institute of Social Research of the University of Michigan (see Institute of Social Research, 1985); it was used as the data source for estimation of the coefficients by Nakamura and Nakamura, and is used as the data source for the decision unit sample for this work's simulation experiment. The dummy variable for race (#10) is classified as a family characteristic because in the PSID race is recorded on the family record, not on the individual record. Not all of the 20 variables in Table 2.1 are used as explanatory variables in each of the three steps of the model; Table 2.2 shows which explanatory variables are used in each step of the model.

The constant terms and the coefficients for the explanatory variables in the three model steps are taken from Tables A.1 through A.3 of Nakamura and Nakamura (1985a). These values were estimated using a data set covering the period 1969 through 1978, selected from the PSID; due to the structure of the PSID, the 1968 through 1979 waves were needed to capture the data for the calendar years 1969 through 1978. A total of 546 women who were from 21 to 64 years old, married, and for whom all data are available throughout the entire period, were found by Nakamura and Nakamura. From that group, 364 wives were selected at random and used

Table 2.1 - Model Variable Definition and Classification

i	Definition (in year t unless otherwise noted)	Type
1	Log of hours of work in t - 1	I
2	Log of hourly wage rate in t - 1 (1967$)	I
3	Proportion of years worked since 18 years of age	I
4	Dummy = 1 if never worked since 18 years of age; = 0 otherwise	I
5	Dummy = 1 if wife has a baby in t ; = 0 otherwise	F
6	Dummy = 1 if youngest child is less than 6 but not a new baby; = 0 otherwise	F
7	Number of children younger than 18 living at home	F
8	Age	I
9	Education	I
10	Dummy = 1 if wife is black; = 0 otherwise	F
11	Earned income of husband (1000's of 1967$)	F
12	Difference between earned income of husband in t and t - 1 (1000's of 1967$)	F
13	Difference between earned income of husband in t and t - 1 if difference negative (1000's of 1967$); = 0 otherwise	F
14	State of residence average hourly wage in manufacturing (1967$)	M
15	Difference between state of residence average hourly wage in manufacturing in t and t - 1 (1967$)	M
16	State of residence unemployment rate	M
17	Difference between state of residence unemployment rate in t and t - 1	M
18	Selection bias term (λ)	I
19	Predicted log of hourly wage (1967$)	I
20	Predicted difference between log of hourly wage in t and t - 1 (1967$)	I

Note: For Type: I = individual; F = family; and M = macroeconomic.
Source: Nakamura and Nakamura (1985a, Tables A1 - A3).

for estimation of the coefficients; the remainder of the data set was used to conduct the out-of-sample simulation experiments reported in that article. The data set was divided into four strata, based on the cross-classification of

Table 2.2 - Explanatory Variable Usage

	Model Step				Model Step		
Var. #	Probit Index	Wage Rate	Hours Worked	Var. #	Probit Index	Wage Rate	Hours Worked
1	X			11	X		X
2	X			12	X		X
3	X	X		13	X		X
4	X	X		14	X	X	
5	X		X	15	X	X	
6	X		X	16	X	X	
7	X		X	17	X	X	
8	X	X	X	18		X	X
9	X	X		19			X
10	X	X		20			X

the wives on two age categories in the current year (under 47 years, or at least 47 years) and two work experience categories in the preceding year (idle, or some work). Tables 2.3 through 2.5 contain the estimated coefficients for the three model steps, respectively.

In the Probit Index step, the dependent variable is the index for the wife's probability of working at any time during the year, called the *probit index*, ϕ. The probability that the wife works during the year is the percentile of the standard normal distribution corresponding to ϕ.

$$\text{probability of working} = \int_{-\infty}^{\phi} \frac{1}{\sqrt{2\pi}} exp\left(- \frac{z^2}{2} \right) dz \,. \tag{2.1}$$

The probit index is modelled as a linear function of a constant term and the explanatory variables indicated in Table 2.2. In the simulation experiment, a wife's probability of working in a year is estimated by this function, and a Monte Carlo determination of the wife's participation in the labor force is

Table 2.3
Estimated Coefficients for Probit Index Step

	Worked in $t - 1$		Idle in $t - 1$	
i	<47	≥47	<47	≥47
constant	0.345	-1.984	0.530	1.997
1	0.289	0.569	0	0
2	0.406	0.258	0	0
3	-0.015	0.442	0.554	1.303
4	0	0	-1.401	-0.795
5	-0.272	0	-1.332	0
6	0.335	0	-0.290	0
7	0.027	0.153	0.036	0.010
8	0.017	0.002	-0.035	-0.047
9	-0.008	-0.001	0.021	0.046
10	-0.217	-0.286	0.357	-0.326
11	0.006	0.020	-0.022	0.220
12	-0.016	0	-0.018	0
13	0	-0.005	0	0.097
14	-0.035	-0.116	0.126	-0.360
15	1.317	2.754	1.167	3.748
16	-0.230	-0.108	-0.050	-0.054
17	0.118	0.055	-0.016	0.053

Source: Nakamura and Nakamura (1985a, Table A1).

made by comparing her deterministic probability of working to a random selection from the uniform(0,1) distribution. If the realized value of this random variable exceeds the wife's probability of working, she remains idle for the entire year and has zero earnings; if not, she enters the labor force for that year and proceeds through the remaining two steps to determine her annual earnings.

To illustrate the simulation model, a wife is selected from the decision unit sample, discussed in Section 2.2, and her performance through the model is calculated at each step. The selected wife is a 41 year

Table 2.4
Estimated Coefficients for Wage Rate Step

	Worked in *t - 1*		Idle in *t - 1*	
i	<47	≥47	<47	≥47
constant	0.111	0.165	-0.854	3.262
3	0.016	-0.028	0.408	2.287
4	0	0	-0.919	-2.008
8	-0.000	-0.001	-0.10	-0.087
9	0.001	-0.002	0.048	0.162
10	-0.011	0.006	0.328	-2.151
14	0.050	-0.054	0.116	-1.288
15	0.311	0.533	0	0
16	-0.058	0.018	0.007	0.043
17	0.002	0.003	0	0
18	1.252	-0.494	0.807	2.508

Source: Nakamura and Nakamura (1985a, Table A2).

Table 2.5
Estimated Coefficients for Hours Worked Step

	Worked in *t - 1*		Idle in *t - 1*	
i	<47	≥47	<47	≥47
constant	-0.193	-0.081	6.714	7.290
5	-0.215	0	0.553	0
6	0.058	0	-0.078	0
7	0.006	0.042	0.050	0.047
8	0.003	-0.003	-0.002	-0.040
11	0.002	0.014	-0.052	-0.012
12	-0.002	0	0	0
13	0	0.025	0	0
18	1.115	1.578	-0.163	0.337
19	0	0	0.033	-0.769
20	1.281	-1.338	0	0

Note: Variable sequence is different than in source.
Source: Nakamura and Nakamura (1985a, Table A3).

old, college graduate, Spanish-American, living in Florida. She has two children, ages 16 and 17 years in 1977. Since the age of 18 years, she has worked 4 years, including 1977 when she worked 1,575 hours and earned $7,200. In 1977, her husband earned $99,999 and in 1978 he earned $72,000. The wife's descriptive characteristics are presented numerically in Table 2.6 as PSID variable values in the second column and the simulation program variable values in the fourth column. Using the simulation program variables indicated in Table 2.2 for the Probit Index step, the program variable values given in column 4 of Table 2.6, and the coefficients

Table 2.6 - Example Wife Variable Values

PSID #	Value	Program Var. #	Value
5203	9	1	7.3620
5703	9	2	0.9237
5743	1575	3	0.1667
5788	7200	4	0
6123	4	5	0
5353	2	6	0
5853	1	7	1
5854	17	8	41
5852	41	9	16
6116	16	10	0
6209	3	11	36.8475
6174	99999	12	-18.2484
6767	72000	13	-18.2484
		14	3.5872
		15	0.0386
		16	4.0
		17	0.0
		18	0.0050
		19	0
		20	0.0953
		YOUNG	.TRUE.
		WORKED	.TRUE.

given in the second column of Table 2.3 for the *worked/young* stratum, this wife's probit index equals 2.9595. Her probability of working in simulated 1978 is 0.9985; this probability would be compared to a random selection from a uniform(0,1) distribution to determine if she would be simulated as working in 1978. It is assumed for this example that she would be simulated to work.

When a wife works during a year, a function of her probit index for the year is used as an explanatory variable for the Wage Rate and Hours Worked steps, variable #18 in Tables 2.1 and 2.2. This function, referred to as the selection bias term, λ, is calculated by

$$\lambda = \frac{f(\phi)}{F(\phi)}, \qquad (2.2)$$

where f denotes the standard normal density function and F denotes the standard normal cumulative distribution function. Heckman (1979) proposed the selection bias term as a simple consistent estimation method for the explanatory variables which when omitted from a regression analysis, due to using censored samples to estimate behavioral models, give rise to specification error. The PSID information on wage rates and hours worked is a censored sample since it does not contain information on the asking wages of those who do not work. Only those whose offered wage, evaluated at zero hours of work, exceeds their asking wage enter the labor force. In the simulation model, the Wage Rate and Hours Worked steps are performed for those wives who have been simulated as entering the labor force in the Probit Index step; thus, their simulated wage rate must exceed their asking wage rate.

For the example wife, with a probit index of 2.9595, her selection bias term is calculated

$$\lambda = \frac{f(2.9595)}{F(2.9595)} = \frac{0.0050}{0.9985} = 0.0050 \,,$$

which is used for the value for X(18) in column 4 of Table 2.6 for the Wage Rate and Hours Worked steps.

In the Wage Rate step, the wife's dependent variable is the log of her wage rate if she was idle during the preceding year, or it is the difference in the logs of the wage rates between the current and preceding years if she had worked during the preceding year. The expected value of this dependent variable is calculated by a linear function of a constant term and the explanatory variables indicated in Table 2.2. The actual value of the dependent variable is stochastically determined by adding a zero-mean normal random variable disturbance term to the expected value. The standard deviation for these distributions, and for those in the Hours Worked step, are not given in the Nakamura paper; however, estimation of these values are described in Section 2.3.

For the example wife, using the simulation program variables indicated in Table 2.2 for the Wage Rate step, the program variable values given in column 4 of Table 2.6, and the coefficients given in the second column of Table 2.4 for the *worked/young* stratum, the expected value of the difference in the log of this wife's 1978 wage rate from her 1977 wage rate equals 0.0953, which is used for variable X(20) in column 4 of Table 2.6 in the Hours Worked step. For this example, the random error term is omitted. Thus the wife's wage rate for simulated 1978 is

$$\text{WAGE}_{78} = exp\left[0.9237 + 0.0953\right] = 2.7704 \,,$$

where the log of her wage rate in 1977 is from X(2) of Table 2.6

A similar procedure is used in the Hours Worked step. The wife's dependent variable is the log of her hours worked if she was idle during the preceding year, or it is the difference in the logs of the hours worked between the current and preceding years if she had worked during the preceding year. The expected value of this dependent variable is calculated by a linear function of a constant and the explanatory variables indicated in Table 2.2. Again, the actual value is stochastically determined by adding a zero-mean normal random variable disturbance term to the expected value.

For the example wife, using the simulation program variables indicated in Table 2.2 for the Hours Worked step, the program variable values given in column 4 of Table 2.6, and the coefficients given in the second column of Table 2.5 for the *worked/young* stratum, the expected value of the difference in the log of this wife's 1978 hours worked from her 1977 hours worked equals 0.1739. Again, the random error term is omitted. Thus the wife's hours worked for simulated 1978 is

$$\text{HOURS}_{78} = exp\left[7.3620 + 0.1739\right] = 1874.1303\,,$$

where the log of hours worked in 1977 is from X(1) of Table 2.6

The wife's annual earnings are calculated by multiplying the wage rate by the number of hours worked during the year;

$$\text{EARNINGS}_{78} = \text{WAGE}_{78} \times \text{HOURS}_{78}$$

$$= 2.7704 \times 1874.1303 = 5192.0906\,,$$

in 1967 dollars.

The two-way random effects model is

$$Y_{ij} = \psi + R_i + C_j + E_{ij}, \tag{2.3}$$

where Y_{ij} is the measurement on the characteristic of interest for the i^{th} decision unit in the j^{th} replication. To put the Nakamura model into the context of this metamodel, the following structure is used. The wives are the decision units, which constitute the effects in the row, i, dimension. Independent replications of the model constitute the effects in the column, j, dimension. The dependent variable, Y_{ij}, is the annual earnings for the i^{th} wife in the j^{th} replication. Each replication of the model produces a column vector of observed values for the annual earnings of the I wives in the decision unit sample. When the model is replicated J times, an $I \times J$ matrix of observed values of Y_{ij} are obtained. Each replication of the model may be thought of as running the economy over the same time period starting at the same initial state, but with different random shocks applied to it. The overall average annual earnings for all wives over all possible replications is represented by the parameter ψ. The row effect for each wife, R_i, is attributable to her deviation from the overall average annual earnings; this effect persists for her over all replications of the economy. The column effect for each replication, C_j, is attributable to the replication's deviation from the overall average annual earnings; this effect has the same affect on all wives in each replication. And the error terms, E_{ij}, are the deviations from the overall average earnings affecting each individual wife on each individual replication of the model.

The two-way random effects model can be a useful supplement to the microsimulation model for policy analysis because it separates the variation contributions in annual earnings. In this sense, the row effects, R_i, can be considered to represent an individual wife's earnings level with respect to others in the labor market; these effects may be of interest to a model user interested in exploring the impact of programs designed to influence an individual's earnings' capability. The standard deviation of the row effects, σ_R, is a measure of the variability of annual earnings among all of the wives. The column effects, C_j, can be considered as the relative state of the economy for each replication, which affects all participants equally; these effects may be of interest to a model user interested in exploring the impact of macroeconomic programs designed to influence the overall state of the economy. The standard deviation of the column effects, σ_C, is a measure of the variability of annual earnings among all of the replications; this is a measure of the variability associated with the behavior of the economy as a stochastic process. The standard deviation of the error term, σ_E, is a measure of the variability of annual earnings across the entire model due to errors arising from operating characteristics estimation, imputation of missing data, and model specification errors.

2.2 The Decision Unit Sample

Since this work is intended to demonstrate output analysis, the experiment was designed to simulate the wives for a single year, 1978, based on their characteristics as of 1977. Nakamura and Nakamura (1985a) did not provide enough information to reconstruct its set of 546 wives. The PSID tapes are periodically updated and corrected, and without having the same tape records available it was not possible to duplicate that set of 546 wives from currently available PSID tapes. For this experiment, a decision unit sample was constructed using the current versions of the PSID tapes. The coefficient estimates for the three simulation model steps as given in Nakamura and Nakamura (1985a) are used in the simulation model computer program, as described in Section 2.3.

In order to obtain the information necessary to simulate the year 1978 based on the wives' characteristics as of 1977, the records from the 1977 through 1979 interviewing years of the PSID were used. The information was obtained from the 1968-1987 Family Level tape, Wave XX. In June, 1991, this wave was recorded on public tapes 157-LISP-167 and 157-CUSP-167 at the University of Michigan Computing Center. On the tapes, the PSID is organized in OSIRIS data files (see Institute for Social Research, 1981). All eligible families with no missing values for the needed variables are used. Eligible families are those from the *SRC subsample* portion of the PSID, for which the wife was between the ages of 29 and 63 years in 1978, and the head and spouse remained married to each other for 1977 and 1978.

Table 2.7 lists the PSID variables used for this model; the numbers refer to the PSID variable identification numbers. The variables in Group A are used to screen for eligible families from among all family records in the tape file; these variables are not saved after the screening is completed. The variables in Group B are used in the simulation computer program; some of these are used directly as explanatory variable values while others are used as arguments in transformation functions to determine other explanatory variable values. The variables in Group C are used to estimate the standard deviations for the stochastic disturbance terms for the Wage Rate and Hours Worked steps, as described in Section 2.3. In fact, these last two PSID variables are the actual 1978 PSID values for the hours worked and wage rate which the model is written to simulate.

Appendix A, Section 1, contains a list of OSIRIS commands used to read the desired information from the tapes, and a list of MIDAS (Fox and

Table 2.7 - PSID Variables Identification

Description	PSID #
Group A	
1968 Interview Number, 1977	5336
Marital Status of Head, Present Status, 1977	5650
Marital Status of Head, Year-to-Year Change, 1978	6219
Marital Status of Head, Year-to-Year Change, 1979	6812
Group B	
Location Measures, State and County, current, state, 1977	5203
Location Measures, State and County, current, state, 1978	5703
Hours, Work, annual, wife, 1978 (lagged one year)	5743
Income, Labor, wife, total, 1978 (lagged one year)	5788
Work History, Years Worked Since 18 (Number of), wife, 1978	6123
Children, Number of, in family unit, total, from birth-17, 1977	5353
Children, Number of, in family unit, total, from birth-17, 1978	5853
Children, Age, youngest in family unit, 1978	5854
Age, Wife, 1978	5852
Education, Head and Wife, grades completed, wife, 1978	6116
Race, 1978	6209
Income, Labor, head, total, 1978 (lagged one year)	6174
Income, Labor, head, total, 1979 (lagged one year)	6767
Group C	
Hours, Work, annual, wife, 1979 (lagged one year)	6348
Income, Labor, wife, total, 1979 (lagged one year)	6398

Source: Institute for Social Research (1985).

Guire, 1976) commands used to eliminate records with missing data and write the valid observations to a data file. All computer work was performed on the University of Michigan's mainframe system with an IBM ES/9021 Model 270 computer. This sequence of commands produces a MIDAS INTERNAL file, which contains 1124 cases for 15 variables (the last 15 variables listed in Table 2.7). Appendix A, Section 2, contains a list of commands used to read the PSID data from the MIDAS INTERNAL file and write the 13 variables needed for the decision unit sample into a FORTRAN formatted file used as input to the computer program written to perform the simulation experiment.

2.3 The Simulation Model Computer Program

A computer program implementing the Nakamura model is written in the FORTRAN programming language, incorporating selected subroutines from the International Mathematical and Statistical Libraries (IMSL, Inc., 1987a,b); a complete listing of the program is given in Appendix B.

In addition to the coefficients given in Tables 2.3 through 2.5, which are set in the computer program in subroutine MODVAL, the Nakamura model uses the average hourly wage in manufacturing and the unemployment rate in the state of residence for each wife; a price deflator is also needed since all dollar values are expressed as 1967 dollars. These macroeconomic characteristic values were gathered from various federal government reports, and set in the computer program in subroutine MACROV. The information on state unemployment rates and the Consumer Price Index is taken from the Handbook of Labor Statistics, Tables 45 and 134 respectively (U. S. Department of Labor, 1980); the state average wage rates in manufacturing are taken from the Handbook of Labor Statistics, Table 90 (U. S. Department of Labor, 1989). The macroeconomic characteristics for the states are listed in Table 2.8; in the PSID the state index numbers are assigned #1 to #49 for the 48 contiguous states and the District of Columbia arranged alphabetically, Alaska is assigned #50, and Hawaii is assigned #51. The Consumer Price Index for 1977 and for 1978 are listed in Table 2.9.Kansas is the only state for which

Table 2.8 - State Macroeconomic Variables

		Unemployment Rate		Manufacturing Avg. Wage Rate	
i	State	1977	1978	1977	1978
1	Alabama	7.4	6.3	4.89	5.40
2	Arizona	8.2	6.1	5.55	6.03
3	Arkansas	6.6	6.3	4.30	4.72
4	California	8.2	7.1	6.00	6.43
5	Connecticut	7.0	5.2	5.56	5.96
6	Colorado	6.2	5.5	5.80	6.21
7	Delaware	8.4	7.6	5.94	6.58
8	Dist. of Columbia	9.7	8.5	5.50	6.72
9	Florida	8.2	6.6	4.63	5.07
10	Georgia	6.9	5.7	4.46	4.88
11	Idaho	5.9	5.7	5.82	6.53
12	Illinois	6.2	6.1	6.28	6.76
13	Indiana	5.7	5.7	6.60	7.17
14	Iowa	4.0	4.0	6.43	7.00
15	Kansas	4.1	3.1	5.11 *	5.64 *
16	Kentucky	4.7	5.2	5.69	6.26
17	Louisiana	7.0	7.0	5.75	6.42
18	Maine	8.4	6.1	4.52	4.91
19	Maryland	6.1	5.6	6.05	6.46
20	Massachusetts	8.1	6.1	5.13	5.54
21	Michigan	8.2	6.9	7.54	8.13
22	Minnesota	5.1	3.8	5.97	6.44
23	Mississippi	7.4	7.1	4.15	4.56
24	Missouri	5.9	5.0	5.75	6.21
25	Montana	6.4	6.0	6.53	7.81
26	Nebraska	3.7	2.9	5.39	5.83
27	Nevada	7.0	4.4	6.10	6.54
28	New Hampshire	5.9	3.8	4.56	4.93
29	New Jersey	9.4	7.2	5.80	6.20
30	New Mexico	7.8	5.8	4.43	4.79
31	New York	9.1	7.7	5.67	6.08

Table 2.8 - State Macroeconomic Variables, cont'd.

		Unemployment Rate		Manufacturing Avg. Wage Rate	
i	State	1977	1978	1977	1978
32	North Carolina	5.9	4.3	4.10	4.47
33	North Dakota	4.8	4.6	5.19	5.55
34	Ohio	6.5	5.4	6.74	7.29
35	Oklahoma	5.0	3.9	5.31	5.81
36	Oregon	7.4	6.0	6.67	7.23
37	Pennsylvania	7.7	6.9	5.85	6.37
38	Rhode Island	8.6	6.6	4.39	4.71
39	South Carolina	7.2	5.7	4.28	4.66
40	South Dakota	3.3	3.1	4.84	5.19
41	Tennessee	6.3	5.8	4.68	5.13
42	Texas	5.3	4.8	5.42	5.88
43	Utah	5.3	3.8	5.18	5.68
44	Vermont	7.0	5.7	4.70	5.10
45	Virginia	5.3	5.4	4.69	5.11
46	Washington	8.8	6.8	6.83	7.56
47	West Virginia	7.1	6.3	6.06	6.68
48	Wisconsin	4.9	5.1	6.16	6.69
49	Wyoming	3.6	3.3	5.70	6.18
50	Alaska	9.4	11.2	9.12	8.86
51	Hawaii	7.3	7.7	5.51	5.90

* Wage rates for Kansas are imputed.
Sources: U. S. Department of Labor (1980, Table 45; 1989, Table 90).

Table 2.9
Consumer Price Index

Year	CPI
1977	181.5
1978	195.4

Source: U. S. Department of Labor (1980, Table 134).

average hourly wage rates in manufacturing are not available for 1977 and 1978. The missing information was estimated by fitting a linear trend model over the years for which the rates were available, 1979 through 1988, using the rate in Nebraska as the explanatory variable and the rate in Kansas as the dependent variable. The imputed rate in Kansas for 1977 is the point estimate from this model using the Nebraska rate for 1977 as the explanatory variable; a similar estimate was obtained for 1978. Nebraska was selected as the source for the explanatory variables since it had the highest coefficient of determination with Kansas ($R^2 = 0.9873$) from among the other states located in Region VII, defined in the Handbook of Labor Statistics, Table 97 (U. S. Department of Labor, 1989). Region VII consists of the states of Iowa, Kansas, Missouri, and Nebraska.

As described in Section 2.1, stochastic disturbance terms are used in the Wage Rate and Hours Worked steps. These disturbance terms are assumed to follow a normal distribution with means equal to zero. The Nakamura article did not report the standard deviations resulting from fitting the models to their data set when the coefficients were estimated; however, the article does report R^2 values for each of the three model steps over each of the four strata. These R^2 values are used with the standard deviations of the actual 1978 wage rates and hours worked from the decision unit sample of wives to estimate the missing standard deviations for the stochastic disturbance terms; details are presented in Appendix C.

The computer program is outlined in Table 2.10. The program is run once for each replication of the model. An initial seed value for the pseudo-random number generators is required as input; at the end of each replication the ending seed value can be written to a file for use as the input value for the subsequent replication. All pseudo-random numbers that may be needed in the run are generated at the outset for efficiency, using IMSL subroutines, in program step 1. The program loops through a series

Table 2.10 - Simulation Program Outline

Step	Description
1	Calculate 3 vectors of pseudo-random numbers
	Start loop, for each wife
2	Read PSID values
3	Transform PSID values to explanatory variable values
4	Determine stratum
5	Calculate probit index
6	Calculate probability of working
7	Monte Carlo determination of working:
	if idle, earnings = 0, end loop for wife;
	if working, continue
8	Calculate selection bias term
9	Calculate wage rate
10	Calculate hours worked
11	Calculate annual earnings
	End loop on wife
12	Report results

of operations for each wife in the decision unit sample. The wife's individual and family characteristics are read from a file containing PSID values in program step 2. The explanatory variable value assignments are made in program step 3. Some of the explanatory variable values are one-to-one transformations of the PSID or macroeconomic values; for examples, age of the wife, or state unemployment rate. However, some program explanatory variables are transformations of the PSID or macroeconomic values; for examples, race (a multilevel categorical variable in the PSID) becomes a single dummy variable, husband's earnings in thousands of 1967 dollars, or the difference between state average wage rates in manufacturing between the current and preceding years in 1967 dollars. The wife's stratum is determined in program step 4. The Probit Index step of the model comprises program steps 5 through 8. If the wife is selected to be working during the year, the Wage Rate step, Hours Worked step and calculation of annual earnings are performed in program steps 9 through 11. After all wives in the decision unit sample have been processed, the vector of annual earnings is reported (program step 12).

2.4 The Simulation Experiment

The simulation experiment consists of 1000 replications of the model. While the number of replications used may be considered large, it is not intended to resolve the issue of sample size for replications. The appropriate number of replications for a simulation experiment depends on many issues including the user's desired confidence level and precision of results, as well as the analysis method used for the metamodel. Sample size issues for simulation experiments are appropriate topics for further research. The seed value used for the pseudo-random number generators in this experiment is 0578143136.

To illustrate the performance of the simulation model, the output from the first ten replications of the model is described in some detail. The output data consists of a 1124×10 matrix of observations on annual earnings. A relative frequency histogram of annual earnings for the entire set of 11240 observations is presented in Figure 2.1. The values on the horizontal axis are the upper bounds on the class intervals. The first class represents the proportion (3091 of 11240) of wife-replications with zero earnings. The rightmost class is open ended; there are 755 observations above $15,000 with the maximum value at $532,827.

Considering the individual replications, descriptive statistics are presented in Table 2.11. The second and third columns describe annual

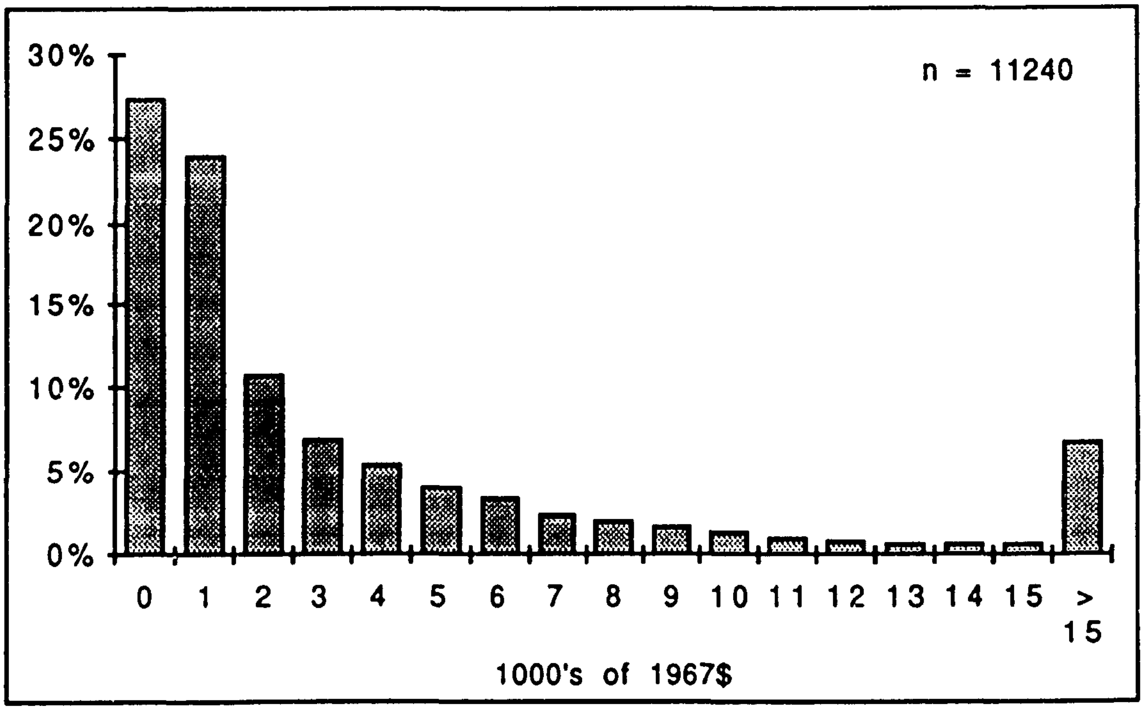

Figure 2.1 - Annual Earnings, All Wives Over All Replications

Table 2.11

Descriptive Statistics of Annual Earnings

	All Wives			Working Wives		
Repl. #	Mean	St.Dev.	# Idle	#	Mean	St.Dev.
1	5007.2	21551.	318	806	6982.7	25182.
2	5143.9	17558.	295	829	6974.4	20133.
3	4617.3	12761.	321	803	6463.1	14699.
4	5033.0	20984.	317	807	7010.0	24488.
5	4975.9	20282.	295	829	6746.5	23366.
6	4803.4	16579.	308	816	6616.4	19150.
7	4667.2	13451.	302	822	6381.9	15379.
8	4280.9	9677.	314	810	5940.4	10960.
9	4384.1	12204.	301	823	5987.5	13924.
10	4925.7	13394.	320	804	6886.2	15407.
Mean	4783.9	15844.1	309.1	814.9	6598.9	18268.8
St.Dev.	289.3	4145.7	10.2	10.2	399.3	4924.8

earnings for all 1124 wives in each replication. The last three columns describe annual earnings of the wives who had worked in each replication. The proportion of wives idle in a replication ranges from 26.2% to 28.5%, with a mean of 27.5%.

The working/idle behavior of the wives across the replications is depicted in Table 2.12. The second column displays the number of wives who worked during the number of replications given in the first column; for examples, 60 of the wives worked in zero replications, and 463 of the wives worked in all 10 replications. The third and fourth columns separate the wives based upon their actual work experience in 1977. Of the 667 wives who had worked in 1977, 566 have been replicated as working in at least 9 replications in 1978; the replicated work experiences in 1978 for the wives

Table 2.12
Replication of Working

1978	Number of Wives		
# Repl's.		1977 Experience	
Worked	All	Idle	Work
0	60	60	0
1	45	43	2
2	58	56	2
3	49	49	0
4	61	58	3
5	58	53	5
6	45	32	13
7	48	19	29
8	62	15	47
9	175	25	150
10	463	47	416
Total	1124	457	667

who were idle in 1977 are well spread across all numbers of replications from 0 to 10. Figure 2.2 graphically displays the replicated work experiences of all wives, comparable to the second column in the table; the divisions of the bars graphically display the replicated work experiences of the wives, according to their actual work experience in 1977 comparable to the third and fourth columns of the table.

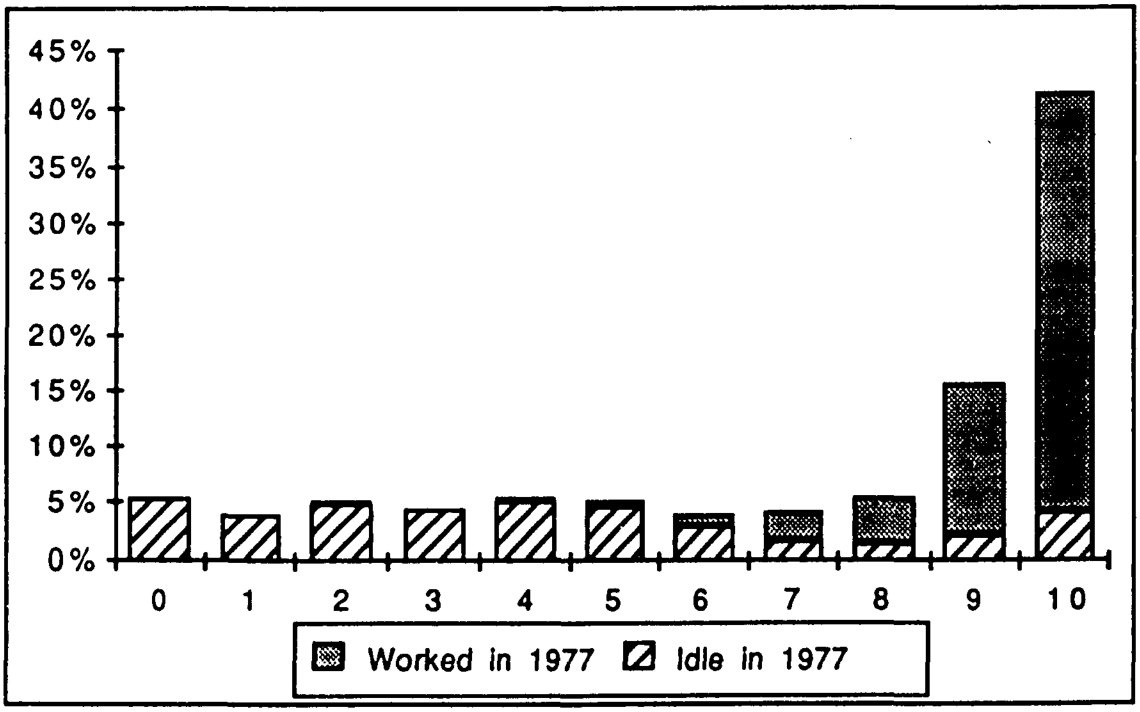

Figure 2.2 - Number of Replications Worked, Out of 10

Relative frequency histograms of the means and standard deviations of annual earnings across replications for each of the 1124 wives are presented in Figures 2.3 and 2.4, respectively. In each of these figures, the values on the horizontal axis are the upper bounds on the class intervals, with the rightmost class being open ended. The 60 wives who were replicated as working in zero replications belong in the zero class for each of these figures. There are 41 wives with mean annual earnings above $20,000 with the maximum at $88,679. There are 83 wives with a standard deviation of annual earnings above $20,000 with the maximum at approximately $171,460.

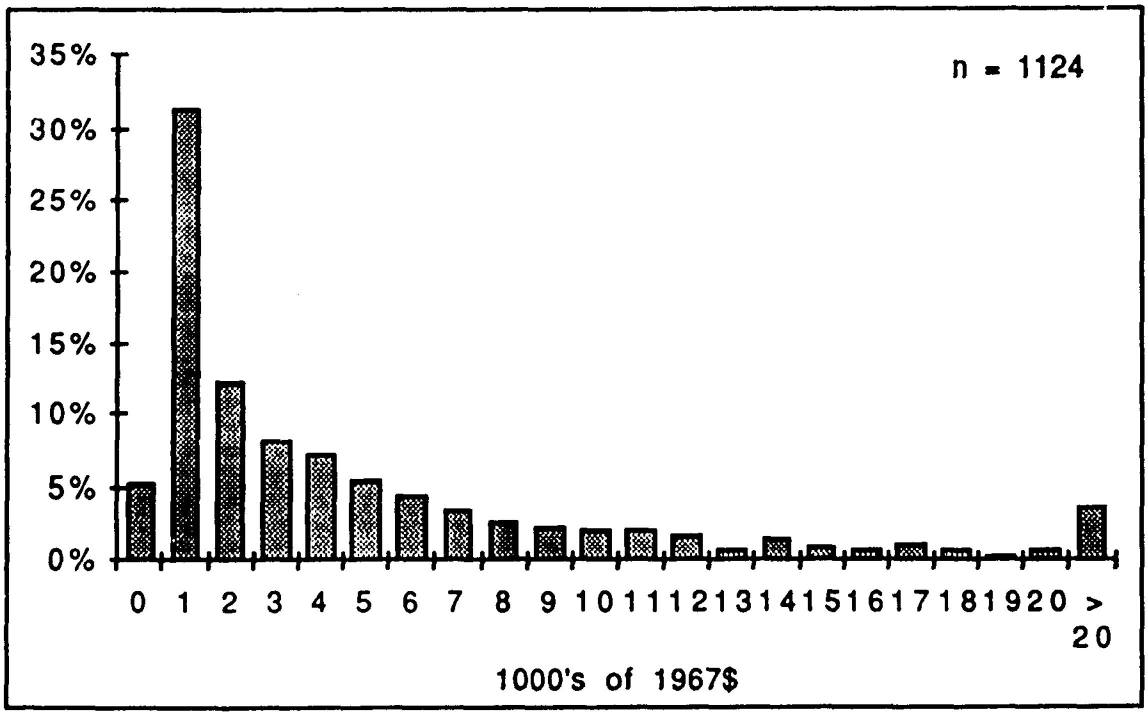

Figure 2.3 - Means of Annual Earnings, Over Replications

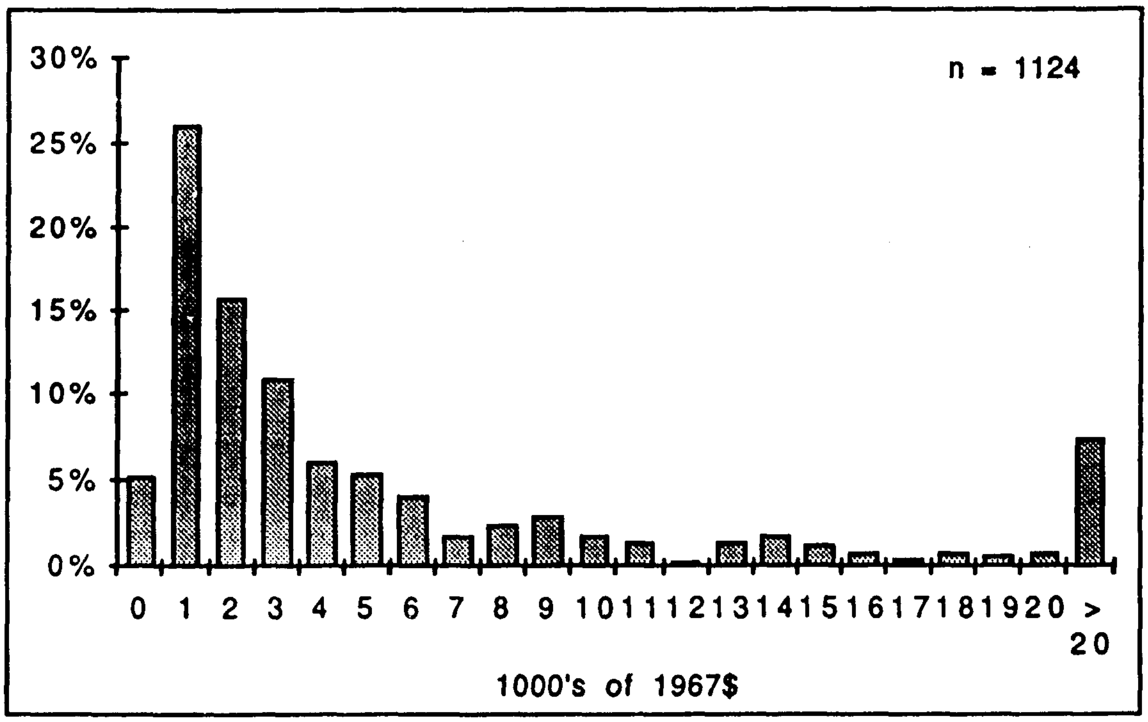

Figure 2.4 - Standard Deviations of Annual Earnings, Over Replications

CHAPTER 3

THE SIMULATION METAMODEL

This chapter presents a general overview of the analysis of the output of the simulation model. Section 3.1 describes the metamodel used. Section 3.2 describes the system performance measure of interest to model users, the mean of a new replication. Methods of determining the distribution of a new column mean are described using sampling theory in Section 3.3, and using Bayesian theory in Section 3.4.

3.1 Two-way Random Effects Model

The balanced two-way random effects model, without interaction, with one observation per cell, and with independent error terms is presented as a metamodel for the analysis of output from repeated, independent replications of a microsimulation model. This model is

$$Y_{ij} = \psi + R_i + C_j + E_{ij}, \tag{3.1}$$

where Y_{ij} is the measurement on the characteristic of interest for the i^{th} decision unit in the j^{th} replication. For the unobserved random variables on the right side, it is assumed that they are statistically independent over all (i, j) and have the following distributions:

the row/decision unit effect, $R_i \sim \text{Normal}\left(0, \sigma_R^2\right)$;

the column/replication effect, $C_j \sim \text{Normal}\left(0, \sigma_C^2\right)$; and,

the error term, $E_{ij} \sim \text{Normal}\left(0, \sigma_E^2\right)$.

There are four parameters: ψ, the overall mean; and $\sigma_R^2, \sigma_C^2, \sigma_E^2$, the row, column, and error variances, respectively.

It is assumed that simple random samples are taken in each of the effects dimensions; that is, a simple random sample of I row effects is selected from the population of all possible row effects, and a simple random sample of J column effects is selected from the population of all possible column effects. It is further assumed that the row and column populations are of infinite size, or, if finite then large enough that it is safe to ignore the effects of sampling from finite populations.

This model is a special case of the two-way random effects model given in Section 6.2 of Box and Tiao (1973, pp. 329 - 340). For this work, it is assumed that no interaction between the decision unit and replication effects occurs, and that there is a single observation for each decision unit in each replication. Table 3.1 presents the analysis of variance formulas, which summarize the sample information in a useful manner, and allow the definition of three sums of squares notation.

The dot subscript notation indicates calculating the arithmetic mean over that dimension:

$$\bar{y}_{j} = \frac{1}{I} \sum_{i} y_{ij},$$

$$\bar{y}_{i.} = \frac{1}{J} \sum_{j} y_{ij} \text{ , and,}$$

Table 3.1 - Analysis of Variance of Two-way Random Effects Model

Source	d.f.	Sum of Squares	E(Mean Square)
Overall mean	1	$IJ\left(\bar{y}_{..} - \psi\right)^2$	$\sigma_E^2 + I\sigma_C^2 + J\sigma_R^2$
Row effect	(I - 1)	$\sum_i J\left(\bar{y}_{i.} - \bar{y}_{..}\right)^2$	$\sigma_E^2 + J\sigma_R^2$
Column effect	(J - 1)	$\sum_j I\left(\bar{y}_{.j} - \bar{y}_{..}\right)^2$	$\sigma_E^2 + I\sigma_C^2$
Residual	(I - 1)(J - 1)	$\sum_i \sum_j \left(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}\right)^2$	σ_E^2
Total	IJ	$\sum_i \sum_j \left(y_{ij} - \psi\right)^2$	

$$\bar{y}_{..} = \frac{1}{IJ} \sum_i \sum_j y_{ij} .$$

The sums of squares shorthand notation follows the standard analysis of variance definitions (Scheffe ,1959, Table 4.2.2, p. 103):

$$SSR = \sum_i J\left(\bar{y}_{i.} - \bar{y}_{..}\right)^2 ,$$

$$SSC = \sum_j I\left(\bar{y}_{.j} - \bar{y}_{..}\right)^2 \text{ , and,}$$

$$SSE = \sum_i \sum_j \left(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}\right)^2 .$$

The row and column dimensions, the sample mean, and the three sums of squares, $\left\{I, J, \bar{y}_{..}, SSR, SSC, SSE\right\}$, constitute a set of sufficient statistics for the two-way random effects model. These sufficient statistics are calculated from a more basic set of sufficient statistics, which has the advantage of eliminating the rounding error associated with the calculation of averages when reporting the sample results:

$$\left\{ I, J, \left(\sum_{i=1}^{I} \sum_{j=1}^{J} y_{ij} \right), \left(\sum_{i=1}^{I} \sum_{j=1}^{J} y_{ij}^{2} \right), \sum_{i=1}^{I} \left(\sum_{j=1}^{J} y_{ij} \right)^{2}, \sum_{j=1}^{J} \left(\sum_{i=1}^{I} y_{ij} \right)^{2} \right\}$$

This set is used as input values in the analytic-numeric approximation computer program discussed in Chapter 6.

3.2 New Column Mean

It is assumed that the model user is interested in the mean of a randomly occurring, as yet unobserved, replication of the model. With sample values being observed for I rows/decision units and J columns/replications, an unobserved replication may, without loss of generality, be referred to as the $(J+1)^{th}$ column. To focus attention on this variable, let X_j denote the mean of the j^{th} replication of the model, where the expectation is taken over the row/decision unit dimension, as given in Equation (1.2):

$$\begin{aligned} X_j &= E_i\left[Y_{ij} \right] \\ &= E_i\left[\psi + R_i + C_j + E_{ij} \right] \\ &= \psi + C_j \, . \end{aligned} \qquad (3.2)$$

Since it is assumed that

$$C_j \sim \text{Normal}\left(0, \sigma_C^2 \right),$$

it follows from Equation (3.2) that

$$X_j \sim \text{Normal}\left(\psi, \sigma_C^2\right).$$

In particular, it is desired to estimate X_{J+1}, the mean of the $(J+1)^{th}$ replication, that is, the mean of any unobserved replication of the model. When the model parameters are unknown, the sample data, and other information in the Bayesian mode of analysis, are used to make inferences about the unknown values. Frequentist theory and Bayesian theory follow different inference approaches.

3.3 Frequentist/Sampling Theory Analysis

The frequentist confidence interval for X_{J+1} is developed by starting with the assumption that all parameter values are known and then removing the assumptions, first for the overall mean, and second for the variances. Let $\{y_{ij}\}$ represent the set of IJ observed values of the y_{ij}'s; and let σ represent the set of variances $\left\{\sigma_R^2, \sigma_C^2, \sigma_E^2\right\}$. A hat (^) over a symbol indicates an estimator of that symbol.

3.3.1 Confidence Interval for X_{J+1}, With ψ and σ Known

Since X_{J+1} follows a normal distribution with

$$E\left(X_{J+1} \mid \psi, \sigma\right) = \psi,$$

and

$$\text{Var}\left(X_{J+1} \mid \psi, \sigma\right) = \sigma_C^2,$$

the $(1 - \alpha)$ confidence interval for X_{J+1} is

$$\psi \pm Z \cdot \left(\sigma_C^2\right)^{1/2}, \tag{3.3}$$

where Z is the value of the $\left(1-\frac{\alpha}{2}\right)^{th}$ percentile from the standard normal random variable distribution.

3.3.2 Confidence Interval for ψ, With σ Known

When ψ is unknown, the sample mean is used as an estimator of the population mean;

$$\hat{\psi} = \bar{y}_{..}\,.$$

From sampling distribution theory, the sample mean follows a normal distribution with

$$E\left(\bar{y}_{..}\right) = \psi,$$

and

$$Var\left(\bar{y}_{..}\right) = \frac{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}{IJ}\,.$$

Thus, a $(1-\alpha)$ confidence interval for ψ, assuming the variances are known, is

$$\bar{y}_{..} \pm Z\cdot\left(\frac{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}{IJ}\right)^{1/2}. \tag{3.4}$$

3.3.3 Confidence Interval for X_{J+1}, With ψ Unknown and σ Known

When the mean parameter, ψ, is unknown, its estimator is used in the definition of X_{J+1}. Let C_{J+1} be the unobserved effect for the $(J+1)^{th}$ replication, then the estimator of the mean of the $(J+1)^{th}$ replication is

$$\hat{X}_{J+1} = \hat{\psi} + C_{J+1}$$

$$= \bar{y}_{..} + C_{J+1} \,. \tag{3.5}$$

The estimator, $\hat{X}_{J+1}$, follows a normal distribution with

$$E\left(\hat{X}_{J+1} \mid \{y_{ij}\}, \sigma\right) = \bar{y}_{..} \,,$$

and, since all replication effects, C_j, are independent,

$$\mathrm{Var}\left(\hat{X}_{J+1} \mid \{y_{ij}\}, \sigma\right) = \mathrm{Var}\left(\bar{y}_{..}\right) + \mathrm{Var}\left(C_{J+1}\right)$$

$$= \frac{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}{IJ} + \sigma_C^2$$

$$= \frac{\sigma_E^2 + J\sigma_R^2 + I(J+1)\sigma_C^2}{IJ} \,.$$

Thus, the confidence interval for X_{J+1}, assuming ψ is unknown and the variances are known, is

$$\bar{y}_{..} \pm Z \cdot \left(\frac{\sigma_E^2 + J\sigma_R^2 + I(J+1)\sigma_C^2}{IJ} \right)^{1/2} \,. \tag{3.6}$$

3.3.4 Confidence Interval for X_{J+1}, With All Parameters Unknown

When the variance parameters are unknown, their estimators are used in the variance of the estimator of the $(J+1)^{th}$ replication mean; the expected value is not affected.

$$E\left(\hat{X}_{J+1} \mid \{y_{ij}\}\right) = \bar{y}_{..} .$$

$$\hat{V}ar\left(\hat{X}_{J+1} \mid \{y_{ij}\}\right) = \frac{\hat{\sigma}_E^2 + J\hat{\sigma}_R^2 + I(J+1)\hat{\sigma}_C^2}{IJ} . \qquad (3.7)$$

Method-of-moments estimators for the variances are found by setting the sample mean squares to their respective expected values. For the error variance,

$$\hat{\sigma}_E^2 = \frac{SSE}{(I-1)(J-1)}$$

$$= MSE . \qquad (3.8)$$

For the column variance,

$$\hat{\sigma}_E^2 + I\hat{\sigma}_C^2 = \frac{SSC}{(J-1)}$$

$$= MSC \text{ , so}$$

$$\hat{\sigma}_C^2 = \frac{MSC - MSE}{I} \qquad (3.9)$$

And for the row variance,

$$\hat{\sigma}_E^2 + J\hat{\sigma}_R^2 = \frac{SSR}{(I - 1)}$$

$$= MSR \text{ , so}$$

$$\hat{\sigma}_R^2 = \frac{MSR - MSE}{J} \quad . \tag{3.10}$$

Equations (3.9) and (3.10) show how the row and column variances may have negative estimates; the error variance estimate is always positive, from Equation (3.8). The mean squares values are functions of the sample data; when MSE > MSC the column variance estimate is negative, and when MSE > MSR the row column variance estimate is negative. The most common way of dealing with the negative estimates is to use the value zero rather than the negative estimate. So, the numerator of the estimated variance of the mean of X_{J+1}, Equation (3.7), may have either the row variance estimate or the column variance estimate or both replaced by zero.

When all variance estimates are positive The numerator of Equation (3.7), the estimator of the variance of X_{J+1}, is

$$\hat{\sigma}_E^2 + J\hat{\sigma}_R^2 + I(J + 1)\hat{\sigma}_C^2 = \frac{SSR}{I - 1} + \frac{(J + 1)SSC}{J - 1} - \frac{(J + 1)SSE}{(I - 1)(J - 1)}$$

$$= \frac{(J - 1)SSR + (I - 1)(J + 1)SSC - (J + 1)SSE}{(I - 1)(J - 1)} \quad . \tag{3.11}$$

Thus, a $(1 - \alpha)$ confidence interval for X_{J+1}, assuming all parameters are unknown, is

$$\bar{y}_{..} \pm t \cdot \left(\frac{(J - 1)SSR + (I - 1)(J + 1)SSC - (J + 1)SSE}{IJ(I - 1)(J - 1)} \right)^{1/2},$$

where t is the value of the $\left(1 - \frac{\alpha}{2}\right)^{th}$ percentile of the Student's t distribution, with degrees of freedom equal to $(I - 1)(J - 1)$.

<u>When the row variance estimate is negative</u> The numerator of Equation (3.7), the estimator of the variance of X_{J+1}, is obtained by setting

$$\hat{\sigma}_R^2 = 0;$$

so,

$$\hat{\sigma}_E^2 + I(J + 1)\hat{\sigma}_C^2 = \frac{SSE}{(I - 1)(J - 1)} + (J + 1)\left(\frac{SSC}{J - 1} - \frac{SSE}{(I - 1)(J - 1)} \right)$$

$$= \frac{(I - 1)(J + 1)SSC - J \cdot SSE}{(I - 1)(J - 1)}. \tag{3.12}$$

Thus, a $(1 - \alpha)$ confidence interval for X_{J+1}, assuming all parameters are unknown, is

$$\bar{y}_{..} \pm t \cdot \left(\frac{(I - 1)(J + 1)SSC - J \cdot SSE}{IJ(I - 1)(J - 1)} \right)^{1/2}.$$

<u>When the column variance estimate is negative</u> The numerator of Equation (3.7), the estimator of the variance of X_{J+1}, is obtained by setting

$$\hat{\sigma}_C^2 = 0\,;$$

so,

$$\hat{\sigma}_E^2 + J\hat{\sigma}_R^2 = \frac{SSR}{I-1}. \qquad (3.13)$$

Thus, a $(1-\alpha)$ confidence interval for X_{J+1}, assuming all parameters are unknown, is

$$\bar{y}_{..} \pm t \cdot \left(\frac{SSR}{IJ(I-1)} \right)^{1/2}.$$

<u>When both row variance and column variance estimates are negative</u>

The numerator of Equation (3.7), the estimator of the variance of X_{J+1}, is obtained by setting

$$\hat{\sigma}_R^2 = 0\text{, and}$$

$$\hat{\sigma}_C^2 = 0\,;$$

so,

$$\hat{\sigma}_E^2 = \frac{SSE}{(I-1)(J-1)}. \qquad (3.14)$$

Thus, a $(1-\alpha)$ confidence interval for X_{J+1}, assuming all parameters are unknown, is

$$\bar{y}_{..} \pm t \cdot \left(\frac{SSE}{IJ(I-1)(J-1)} \right)^{1/2}.$$

3.4 Bayesian Analysis

The objective is to find the posterior distribution of X_{J+1}. This posterior distribution is determined by integrating, over the model parameters, the product of the conditional distribution of X_{J+1}, given its mean and variance parameters, with the joint posterior distribution of the model parameters.

$$f\left(X_{J+1} \mid \{y_{ij}\}\right)$$

$$\propto \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{0}^{+\infty} \int_{0}^{+\infty} f\left(X_{J+1} \mid \psi, \sigma_C^2\right) \cdot f\left(\psi, \sigma_R^2, \sigma_C^2, \sigma_E^2 \mid \{y_{ij}\}\right) d\sigma_E^2 \, d\sigma_C^2 \, d\sigma_R^2 \, d\psi$$

$$\propto \int_{\Theta} f\left(X_{J+1} \mid \theta\right) \cdot f\left(\theta \mid \{y_{ij}\}\right) d\theta \, ,$$

where θ denotes the set of model parameters $\left\{\psi, \sigma_R^2, \sigma_C^2, \sigma_E^2\right\}$, and Θ represents the domains of integration for θ. Note that the conditional distribution of X_{J+1}, given its mean and variance parameters, is independent of the row and error variances and of the data; so

$$f\left(X_{J+1} \mid \psi, \sigma_C^2\right) = f\left(X_{J+1} \mid \psi, \sigma_R^2, \sigma_C^2, \sigma_E^2, \{y_{ij}\}\right), \text{ or}$$

$$f\left(X_{J+1} \mid \theta\right) = f\left(X_{J+1} \mid \theta, \{y_{ij}\}\right).$$

When the posterior distribution of model parameters is factored into the likelihood and prior distributions, Equation (3.8) becomes

$$f\left(X_{J+1} \mid \{y_{ij}\}\right) \propto \int_{\Theta} f\left(X_{J+1} \mid \theta\right) \cdot L\left(\theta \mid \{y_{ij}\}\right) \cdot f\left(\theta\right) d\theta \, .$$

The integration over ψ can be performed analytically, while the integrations over $\left\{\sigma_R^2, \sigma_C^2, \sigma_E^2\right\}$ are not tractable and need to be approximated.

The derivation of the prior and likelihood functions, and integration over the mean parameter, ψ, are presented in Chapter 4; approximations of the integrals over the variances are developed in Chapter 5; implementation of the approximations of the integrals, and presentation of results, are given in Chapter 6.

CHAPTER 4

EXACT ANALYSIS FOR FOR THE POSTERIOR DISTRIBUTION OF THE MEAN OF THE $(J+1)^{TH}$ REPLICATION

This chapter presents analytical results for the Bayesian analysis of the posterior distribution of the mean of a new column of a balanced, two-way random effects model, the mean of the $(J+1)^{th}$ replication of the microsimulation model. The derivation of the likelihood function of the metamodel is presented in section 4.1. Informative and non-informative prior distributions are described in section 4.2. Section 4.3 presents the joint posterior distribution of the metamodel parameters. Section 4.4 presents the posterior distribution of the mean of the $(J+1)^{th}$ replication. The expected value and variance of the mean of the $(J+1)^{th}$ replication are needed for the approximation methods discussed in Chapter 5; their derivations are presented in sections 4.5 and 4.6, respectively. The distributions of the random variables used in this chapter are defined in Appendix E.

4.1 Likelihood Function

As described in Chapter 3, the simulation metamodel is

$$Y_{ij} = \psi + R_i + C_j + E_{ij}, \tag{4.1}$$

where Y_{ij} is the measurement on the characteristic of interest for the i^{th} decision unit in the j^{th} replication. For the random variables on the right side, it is assumed that they are statistically independent over all $\{i, j\}$ and have the following distributions:

$$R_i \sim \text{Normal}\left(0, \sigma_R^2\right);$$

$$C_j \sim \text{Normal}\left(0, \sigma_C^2\right); \text{ and,}$$

$$E_{ij} \sim \text{Normal}\left(0, \sigma_E^2\right).$$

There are four parameters: ψ, the overall mean; and σ_R^2, σ_C^2, σ_E^2, the row, column, and error variances, respectively.

Let $\{y_{ij}\}$ denote the set of IJ observations on the dependent variable, y_{ij}'s; let $\{R_i\}$ denote the set of I decision unit effects, R_i's; let $\{C_j\}$ denote the set of J replication effects, C_j's; $\{E_{ij}\}$ denote the set of IJ error terms, E_{ij}'s. Let θ denote the model parameter set $\left\{\psi, \sigma_R^2, \sigma_C^2, \sigma_E^2\right\}$; and let $L\left(\theta \mid \{y_{ij}\}\right)$ denote the likelihood of the parameter set conditioned on the observed data. Let $g(\cdot)$ denote a general function; and let $f(\cdot)$ denote a probability density function. The *kernel* of a probability density function is the part of the function that changes with the random variable; a kernel is proportional to a probability density function, omitting all constants (see Raiffa and Schlaifer, 1961, p. 30).

The likelihood function of the parameter set is proportional to the probability of observing the data conditioned upon the parameter values;

$$L\left(\theta \mid \{y_{ij}\}\right) \propto f\left(\{y_{ij}\} \mid \theta\right).$$

The right side of this equation is the joint probability of the IJ observed values of individual y_{ij}'s. This joint probability of $\{y_{ij}\}$ is found by integrating the joint probability of $\left(\{y_{ij}\}, \{R_i\}, \{C_j\}\right)$ over the R_i's and C_j's,

$$f\left(\{y_{ij}\} \mid \theta\right) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f\left(\{y_{ij}\}, \{R_i\}, \{C_j\} \mid \theta\right) \prod_{i=1}^{I} dR_i \prod_{j=1}^{J} dC_j \,. \qquad (4.2)$$

The joint probability function may be factored, using the assumption that the R_i and C_j are statistically independent over all i and j,

$$f\left(\{y_{ij}\}, \{R_i\}, \{C_j\} \mid \theta\right) = f\left(\{y_{ij}\} \mid \{R_i\}, \{C_j\}, \theta\right) \cdot f\left(\{R_i\} \mid \theta\right) \cdot f\left(\{C_j\} \mid \theta\right). \qquad (4.3)$$

For the decision unit effects, with mean equal to zero, and independence across decision units,

$$f\left(\{R_i\} \mid \sigma_R^2\right) = \prod_{i=1}^{I} f\left(R_i \mid 0, \sigma_R^2\right)$$

$$= \prod_{i=1}^{I} \left(2\pi\sigma_R^2\right)^{-1/2} exp\left[\frac{-R_i^{\,2}}{2\sigma_R^2}\right]$$

$$= \left(2\pi\sigma_R^2\right)^{-I/2} exp\left[-\sum_{i=1}^{I} \frac{R_i^{\,2}}{2\sigma_R^2}\right]. \qquad (4.4)$$

Similarly, for the replication effects, with mean equal to zero, and independence across replications,

$$f\left(\{C_j\} \mid \sigma_C^2\right) = \left(2\pi\sigma_C^2\right)^{-J/2} exp\left[-\sum_{j=1}^{J} \frac{C_j^{\,2}}{2\sigma_C^2}\right]. \qquad (4.5)$$

For the error terms, with mean equal to zero,

$$f\left(E_{ij} \mid \{R_i\}, \{C_j\}, \theta\right) = \left(2\pi\sigma_E^2\right)^{-1/2} exp\left[-\frac{E_{ij}^{\ 2}}{2\sigma_E^2}\right]. \tag{4.6}$$

Transforming the model in Equation (4.1) gives

$$E_{ij} = Y_{ij} - \psi - R_i - C_j .$$

Incorporating this transformation, with a Jacobian equal to 1, into the normal probability distribution for the random error terms in Equation (4.6) gives

$$f\left(y_{ij} \mid \{R_i\}, \{C_j\}, \theta\right) = \left(2\pi\sigma_E^2\right)^{-1/2} exp\left[-\frac{\left(Y_{ij} - \psi - R_i - C_j\right)^2}{2\sigma_E^2}\right].$$

And, since the $\left(y_{ij} \mid \{R_i\}, \{C_j\}, \theta\right)$ are independent over all i and j,

$$f\left(\{y_{ij}\} \mid \{R_i\}, \{C_j\}, \theta\right) = \left(2\pi\sigma_E^2\right)^{-IJ/2} exp\left[-\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{\left(Y_{ij} - \psi - R_i - C_j\right)^2}{2\sigma_E^2}\right]. \tag{4.7}$$

Substituting Equations (4.4), (4.5), and (4.7) into Equation (4.3) and then into Equation (4.2), gives,

$$f\left(\{y_{ij}\} \mid \theta\right) = \left(2\pi\right)^{-(IJ+I+J)/2}\left(\sigma_R^2\right)^{-I/2}\left(\sigma_C^2\right)^{-J/2}\left(\sigma_E^2\right)^{-IJ/2}$$

$$\times \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g\left(R_i, C_j\right) \prod_{i=1}^{I} dR_i \prod_{j=1}^{J} dC_j ,$$

where

$$g\left(R_i, C_j\right) = exp\left[-\sum_{i=1}^{I} \frac{{R_i}^2}{2\,\sigma_R^2} - \sum_{j=1}^{J} \frac{{C_j}^2}{2\,\sigma_C^2} - \sum_{i=1}^{I}\sum_{j=1}^{J} \frac{\left(y_{ij} - \psi - R_i - C_j\right)^2}{2\,\sigma_E^2}\right].$$

The integrations over the R_i and the C_j are performed analytically in Appendix F.

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g\left(R_i, C_j\right) \prod_{i=1}^{I} dR_i \prod_{j=1}^{J} dC_j$$

$$= \left(2\,\pi\right)^{(I+J)/2} \left(\sigma_R^2\right)^{I/2} \left(\sigma_C^2\right)^{J/2} \left(\sigma_E^2\right)^{(I+J-1)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I-1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times\, exp\left[-\frac{SSE}{2\,\sigma_E^2} - \frac{SSR}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC}{2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{IJ\left(\bar{y}_{..} - \psi\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right].$$

Substituting for the integrals,

$$f\left(\{y_{ij}\}\,|\,\theta\right) = \left(2\,\pi\right)^{-(IJ+I+J)/2}\left(\sigma_R^2\right)^{-I/2}\left(\sigma_C^2\right)^{-J/2}\left(\sigma_E^2\right)^{-IJ/2}$$

$$\times\left(2\,\pi\right)^{(I+J)/2}\left(\sigma_R^2\right)^{I/2}\left(\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(I+J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I-1)/2}$$

$$\times\left(\sigma_E^2+I\sigma_C^2\right)^{-(I-1)/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}$$

$$\times\, exp\left[-\frac{SSE}{2\,\sigma_E^2}-\frac{SSR}{2\left(\sigma_E^2+J\sigma_R^2\right)}-\frac{SSC}{2\left(\sigma_E^2+I\sigma_C^2\right)}-\frac{IJ\left(\bar{y}_{..}-\psi\right)^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$=\left(2\,\pi\right)^{-IJ/2}\left(\sigma_E^2\right)^{-(I-1)(J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I-1)/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}$$

$$\times\, exp\left[-\frac{SSE}{2\,\sigma_E^2}-\frac{SSR}{2\left(\sigma_E^2+J\sigma_R^2\right)}-\frac{SSC}{2\left(\sigma_E^2+I\sigma_C^2\right)}-\frac{IJ\left(\bar{y}_{..}-\psi\right)^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right].$$

Omitting constant terms, the kernel of the likelihood function may be expressed as:

$$L\left(\theta \mid \{y_{ij}\}\right)$$

$$\propto \left(\sigma_E^2\right)^{-(I-1)(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{-(I-1)/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[-\frac{SSE}{2\,\sigma_E^2} - \frac{SSR}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC}{2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{IJ\left(\bar{y}_{..} - \psi\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]. \qquad (4.8)$$

4.2 Prior Distributions For Model Parameters

4.2.1 Informative priors

Conjugate prior distributions for the set of model parameters, $\left\{\psi, \sigma_R^2, \sigma_C^2, \sigma_E^2\right\}$, are used throughout. It is assumed that the row variance and column variance are statistically independent, so the joint prior distribution can be factored as

$$f\left(\psi, \sigma_R^2, \sigma_C^2, \sigma_E^2\right) = f\left(\psi \mid \sigma_R^2, \sigma_C^2, \sigma_E^2\right) \cdot f\left(\sigma_R^2 \mid \sigma_E^2\right) \cdot f\left(\sigma_C^2 \mid \sigma_E^2\right) \cdot f\left(\sigma_E^2\right).$$

The prior distribution for ψ conditional on $\left\{\sigma_R^2, \sigma_C^2, \sigma_E^2\right\}$ is assumed to be normal, with parameters μ and $\dfrac{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}{\tau}$,

$$\left(\psi \mid \sigma_R^2, \sigma_C^2, \sigma_E^2\right) \sim \text{Normal}\left(\mu, \frac{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}{\tau}\right);$$

so,

$$f\left(\psi \mid \sigma_R^2, \sigma_C^2, \sigma_E^2\right) \propto \left(\frac{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}{\tau}\right)^{-1/2} exp\left[\frac{-\left(\psi - \mu\right)^2}{2\left(\frac{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}{\tau}\right)}\right]$$

$$\propto \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} exp\left[\frac{-\tau\left(\psi - \mu\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right].$$

The prior distribution for σ_E^2 is assumed to be Inverse Gamma, with parameters α_E and β_E,

$$\sigma_E^2 \sim \text{Inverse Gamma}\left(\alpha_E, \beta_E\right);$$

so,

$$f\left(\sigma_E^2\right) \propto \left(\sigma_E^2\right)^{-(\alpha_E + 1)} exp\left[\frac{-1}{\sigma_E^2 \beta_E}\right].$$

The prior distribution for σ_R^2 conditional on σ_E^2, and independent of σ_C^2, is assumed to have the following form with parameters α_R and β_R,

$$f\left(\sigma_R^2 \mid \sigma_E^2\right) \propto \left(\sigma_E^2 + J\sigma_R^2\right)^{-(\alpha_R + 1)} exp\left[\frac{-1}{\left(\sigma_E^2 + J\sigma_R^2\right)\beta_R}\right].$$

This is equivalent to assuming that

$$\left(\sigma_E^2 + J\sigma_R^2\right) \sim \text{Inverse Gamma}\left(\alpha_R, \beta_R\right).$$

Similarly, the prior distribution for σ_C^2 conditional on σ_E^2, and independent of σ_R^2, is assumed to have the following form with parameters α_C and β_C,

$$f\left(\sigma_E^2 + I\sigma_C^2\right) \propto \left(\sigma_E^2 + I\sigma_C^2\right)^{-(\alpha_C + 1)} exp\left[\frac{-1}{\left(\sigma_E^2 + I\sigma_C^2\right)\beta_C}\right].$$

This is equivalent to assuming that

$$\left(\sigma_E^2 + I\sigma_C^2\right) \sim \text{Inverse Gamma}\left(\alpha_C, \beta_C\right).$$

Multiplying the above prior distribution kernels gives the kernel of the joint prior distribution.

$$f\left(\psi, \sigma_R^2, \sigma_C^2, \sigma_E^2\right) \propto \left(\frac{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}{\tau}\right)^{-1/2} exp\left[\frac{-\tau\left(\psi - \mu\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times \left(\sigma_E^2 + J\sigma_R^2\right)^{-(\alpha_R + 1)} exp\left[\frac{-1}{\left(\sigma_E^2 + J\sigma_R^2\right)\beta_R}\right]$$

$$\times \left(\sigma_E^2 + I\sigma_C^2\right)^{-(\alpha_C + 1)} exp\left[\frac{-1}{\left(\sigma_E^2 + I\sigma_C^2\right)\beta_C}\right]$$

$$\times \left(\sigma_E^2\right)^{-(\alpha_E + 1)} exp\left[\frac{-1}{\sigma_E^2 \beta_E}\right]$$

$$\propto \left(\sigma_E^2\right)^{-(\alpha_E+1)} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(\alpha_R+1)} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(\alpha_C+1)} \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[-\frac{\beta_E^{-1}}{\sigma_E^2} - \frac{\beta_C^{-1}}{\sigma_E^2 + I\sigma_C^2} - \frac{\beta_R^{-1}}{\sigma_E^2 + J\sigma_R^2} - \frac{\tau\left(\psi - \mu\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]. \tag{4.9}$$

4.2.2 Non-informative priors

The non-informative prior distributions are obtained from the informative prior distributions by taking the prior distribution parameters, τ, α, and β, to their respective limiting values as follows: for the prior distribution of the mean ψ, let $\tau \to 0$; for each of the variance prior distributions, let $\alpha \to 0$ and $\beta \to \infty$. Using these limiting values, the prior distribution for ψ is locally uniform, and the prior distributions for the variance functions, $\log\left(\sigma_E^2 + J\sigma_R^2\right)$, $\log\left(\sigma_E^2 + I\sigma_C^2\right)$ and $\log\left(\sigma_E^2\right)$, are locally uniform.

Applying the limiting values on τ, α, and β to the joint prior distribution in Equation (4.9) gives

$$f\left(\psi, \sigma_R^2, \sigma_C^2, \sigma_E^2\right) \propto \left(\sigma_E^2\right)^{-1} \left(\sigma_E^2 + J\sigma_R^2\right)^{-1} \left(\sigma_E^2 + I\sigma_C^2\right)^{-1}.$$

Note that these priors are improper, in the usual sense that they do not integrate to unity.

4.3 Joint Posterior Distribution of Model Parameters

The joint posterior distribution of model parameters is proportional to the product of the likelihood function and the prior distribution.

$$f\left(\theta \mid \{y_{ij}\}\right) \propto L\left(\theta \mid \{y_{ij}\}\right) \cdot f\left(\theta\right).$$

Multiplying the likelihood function, Equation (4.8), and the joint informative prior distribution, Equation (4.9), gives the joint posterior distribution.

$$f\left(\theta \mid \{y_{ij}\}\right)$$

$$\propto \left(\sigma_E^2\right)^{-(I-1)(J-1)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I-1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[-\frac{SSE}{2\,\sigma_E^2} - \frac{SSR}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC}{2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{IJ\left(\bar{y}_{..} - \psi\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times \left(\sigma_E^2\right)^{-(\alpha_E + 1)} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(\alpha_R + 1)} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(\alpha_C + 1)}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[-\frac{\beta_E^{-1}}{\sigma_E^2} - \frac{\beta_C^{-1}}{\sigma_E^2 + I\sigma_C^2} - \frac{\beta_R^{-1}}{\sigma_E^2 + J\sigma_R^2} - \frac{\tau\left(\psi - \mu\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\propto \left(\sigma_E^2\right)^{-(IJ-I-J+2\alpha_E+3)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I+2\alpha_R+1)/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J+2\alpha_C+1)/2}$$

$$\times\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1} exp\left[-\frac{SSE+2\beta_E^{-1}}{2\sigma_E^2}-\frac{SSR+2\beta_R^{-1}}{2\left(\sigma_E^2+J\sigma_R^2\right)}-\frac{SSC+2\beta_C^{-1}}{2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{IJ\left(\bar{y}_{..}-\psi\right)^2+\tau\left(\psi-\mu\right)^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right].$$

Let σ denote the set of variances, $\left\{\sigma_R^2, \sigma_C^2, \sigma_E^2\right\}$, then $\theta = \left(\psi, \sigma\right)$; and let Σ denote the domain of integration for σ.

$$f\left(\theta \mid \{y_{ij}\}\right) = f\left(\psi, \sigma \mid \{y_{ij}\}\right)$$

$$= C_1\left(\sigma_E^2\right)^{-(IJ-I-J+2\alpha_E+3)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I+2\alpha_R+1)/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J+2\alpha_C+1)/2}$$

$$\times\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1} exp\left[-\frac{SSE+2\beta_E^{-1}}{2\sigma_E^2}-\frac{SSR+2\beta_R^{-1}}{2\left(\sigma_E^2+J\sigma_R^2\right)}-\frac{SSC+2\beta_C^{-1}}{2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{IJ\left(\bar{y}_{..}-\psi\right)^2+\tau\left(\psi-\mu\right)^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right], \qquad (4.10)$$

where C_1 denotes the normalizing constant.

The normalizing constant is defined in Appendix G; after integrating over ψ,

$$C_1^{-1} = \left(\frac{2\pi}{IJ + \tau}\right)^{1/2} \int_{\Sigma} g\left(\sigma \mid \{y_{ij}\}\right) d\sigma \qquad (4.11)$$

where

$$g\left(\sigma \mid \{y_{ij}\}\right) = \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} \right]$$

$$\times exp\left[\frac{IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right]. \qquad (4.12)$$

4.4 Posterior Distribution of the Mean of the (J+1)th Replication

As discussed in Section 2 of Chapter 3, it is assumed that the model user is interested in the mean of a randomly occurring, as yet unobserved, replication of the model which, without loss of generality, may be referred to as the $(J+1)^{th}$ replication. In particular, it is desired to estimate X_{J+1}, the mean of the $(J+1)^{th}$ replication. From Equation (3.2),

$$X_{J+1} = \psi + C_{J+1};$$

and

$$X_{J+1} \sim \text{Normal}\left(\psi, \sigma_C^2\right).$$

The posterior distribution of the mean of the (J+1)th replication is found by integrating, over the model parameters, the joint distribution of the mean of the (J+1)th replication and the model parameters:

$$f\left(X_{J+1} \mid \{y_{ij}\}\right) = \int_{\Sigma} \int_{-\infty}^{+\infty} f\left(X_{J+1}, \psi, \sigma \mid \{y_{ij}\}\right) d\psi \, d\sigma .$$

The integrand is obtained by multiplying the conditional distribution of the mean of the (J+1)th replication by the posterior distribution of the model parameters, Equation (4.10).

$$f\left(X_{J+1}, \psi, \sigma \mid \{y_{ij}\}\right) = f\left(X_{J+1} \mid \psi, \sigma\right) \cdot f\left(\psi, \sigma \mid \{y_{ij}\}\right)$$

$$= \left(2\pi\sigma_C^2\right)^{-1/2} exp\left[-\frac{\left(X_{J+1} - \psi\right)^2}{2\sigma_C^2}\right]$$

$$\times C_1 \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1} exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{IJ\left(\bar{y}_{..} - \psi\right)^2 + \tau\left(\psi - \mu\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$= C_1\left(2\pi\sigma_C^2\right)^{-1/2}\left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2}$$

$$\times\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{\left(X_{J+1} - \psi\right)^2}{2\sigma_C^2} - \frac{IJ\left(\bar{y}_{..} - \psi\right)^2 + \tau\left(\psi - \mu\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right].$$

The integration over ψ is performed analytically in Appendix H, giving the joint distribution of the mean of the $(J+1)^{th}$ replication and the variances.

$$\int_{-\infty}^{+\infty} f\left(X_{J+1}, \psi, \sigma \mid \{y_{ij}\}\right) d\psi = f\left(X_{J+1}, \sigma \mid \{y_{ij}\}\right)$$

$$= C_1 \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} - \frac{\left(X - \dfrac{IJ\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)^2}{2\left(\dfrac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau}\right)}\right].$$

(4.13)

The posterior distribution of the mean of the $(J+1)^{th}$ replication can now be expressed as

$$f\left(X_{J+1} \mid \{y_{ij}\}\right) = \int_{\Sigma} f\left(X_{J+1}, \sigma \mid \{y_{ij}\}\right) d\sigma . \tag{4.14}$$

Analytic solutions for the integrations over the variances, σ, in Equation (4.14) have not been found; approximations based on LaPlace's method are presented in Chapter 5 and implemented in Chapter 6.

4.5 Posterior Expected Value of the Mean of the (J+1)th Replication

The posterior expected value of the mean of the (J+1)th replication is found by taking the expected value of X_{J+1} with respect to its posterior distribution, given in Equation (4.14). The integrations are performed in Appendix I.

$$E\left[X_{J+1} \mid \{y_{ij}\}\right] = \int_{-\infty}^{+\infty} X_{J+1} \cdot f\left(X_{J+1} \mid \{y_{ij}\}\right) dX_{J+1}$$

$$= \int_{-\infty}^{+\infty} \int_{\Sigma} X_{J+1} \cdot f\left(X_{J+1}, \sigma \mid \{y_{ij}\}\right) d\sigma \, dX_{J+1} \,.$$

$$= \frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}, \tag{4.15}$$

a weighted average of the prior mean and the mean of the observations, consistent with DeGroot (1970, Theorem 1, page 196).

4.6 Posterior Variance of the Mean of the (J+1)th Replication

The posterior variance of the mean of the (J+1)th replication is

$$V\left[X_{J+1} \mid \{y_{ij}\}\right] = E\left[X_{J+1}^2 \mid \mathbf{y}\right] - \left(E\left[X_{J+1} \mid \{y_{ij}\}\right]\right)^2$$

$$= E\left[X_{J+1}^2 \mid \mathbf{y}\right] - \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)^2, \tag{4.16}$$

after substituting for the posterior expected value from Equation (4.15). The posterior expected value of X_{J+1}^2 is found by using Equation (4.14). The integrations, to the extent possible, are performed in Appendix J.

$$E\left[X_{J+1}^2 \mid y\right] = \int_{-\infty}^{+\infty} X_{J+1}^2 \cdot f\left(X_{J+1} \mid \{y_{ij}\}\right) dX_{J+1}$$

$$= \int_{-\infty}^{+\infty} \int_{\Sigma} X_{J+1}^2 \cdot f\left(X_{J+1}, \sigma \mid \{y_{ij}\}\right) d\sigma \, dX_{J+1}$$

$$= \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right)^2$$

$$+ \frac{\int_{\Sigma} \left[\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right) g\left(\sigma \mid \{y_{ij}\}\right) \right] d\sigma}{\int_{\Sigma} g\left(\sigma \mid \{y_{ij}\}\right) d\sigma},$$

where $g\left(\sigma \mid \{y_{ij}\}\right)$ is given in Equation (4.12). Substituting into Equation (4.16), the squares of the posterior expected value cancel, giving

$$V\left[X_{J+1} \mid \{y_{ij}\}\right] = \frac{\int_{\Sigma} \left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right) \cdot g\left(\sigma \mid \{y_{ij}\}\right) d\sigma}{\int_{\Sigma} g\left(\sigma \mid \{y_{ij}\}\right) d\sigma}. \quad (4.17)$$

As with the posterior distribution of the mean of the (J+1)th replication in Equation (4.14), analytic solutions for the integrations over the variances, σ, in the numerator and denominator in Equation (4.17) have not been found; approximations based on LaPlace's method are presented in Chapter 5 and implemented in Chapter 6.

CHAPTER 5

APPROXIMATION ANALYSIS FOR THE POSTERIOR DISTRIBUTION OF THE MEAN OF THE (J+1)TH REPLICATION

This chapter presents methods for obtaining approximate values for the integrals in the posterior distribution, and variance, of the mean of the $(J+1)^{th}$ replication of the microsimulation model. Section 5.1 presents the general strategy used in finding approximate values for the intractable integrals from Chapter 4, and gives a general integrand function that includes all cases. Section 5.2 presents the LaPlace method for integral approximation, and four special situations that arise for the two-way random effects model. Sections 5.3 through 5.6 present the LaPlace method applied to the four special situations.

5.1 Approximation Strategy

5.1.1 In general

The posterior distribution of the mean of the $(J+1)^{th}$ replication of a microsimulation model is derived in Chapter 4, given in Equation (4.14) and repeated here.

$$f\left(X_{J+1} \mid \{y_{ij}\}\right) = \int_{\Sigma} f\left(X_{J+1}, \sigma \mid \{y_{ij}\}\right) d\sigma \ . \tag{5.1}$$

The integrand is given in Equation (4,13); it contains a normalizing constant, C_1, whose inverse is given in Equation (4.11) and repeated here;

$$C_1^{-1} = \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} \int_{\Sigma} g\left(\sigma \,|\, \{y_{ij}\}\right) d\sigma .$$

Integrations over the variances thus appear in the numerator and the denominator of the right side of the posterior distribution; however, these integrations are intractable. Approximations for these integrals, based on LaPlace's method, are developed later in this chapter. This method is described in detail in Bruijn (1961), and applied to Bayesian analysis by Leonard (1982) and a number of articles including Kass, Tierney and Kadane (1988), Tierney and Kadane (1986) and Tierney, Kass and Kadane (1987, 1989a, 1989b).

The general idea of LaPlace's method is to approximate the value of the integral by a function of the integrand evaluated at its mode. In some situations the mode can be found algebraically, and the LaPlace approximation can be solved algebraically as a function of the variable of integration. See Tierney and Kadane (1986) for examples involving simpler integrands which permit an algebraic solution for the approximation of the integral. When the integrand is sufficiently complex that the mode cannot be found algebraically, as the case here using the two-way random effects model, the LaPlace method can be used by finding the mode, and evaluating the approximation function, using numerical methods.

Since the modes for the variances in the integrands in Equation (5.1) must be found numerically, the integral can be approximated only for a specific value of X_{J+1}. Consequently, the approximation procedure must be repeated for each point in an appropriate interval for X_{J+1}. This sequence of approximations produces a discrete set of values that approximates, after appropriate re-scaling, the continuous posterior distribution of X_{J+1}. As noted in Tierney, Kass and Kadane (1989a),

> "A weakness of these approximations is that they generally do not integrate to one. Numerical integration has to be used to renormalize the approximations. ... the shape of the marginal density is approximated more accurately than the constant of integration."

The values of X_{J+1} in a symmetric interval around its posterior mean constitute an appropriate set over which to approximate the posterior distribution of X_{J+1}. Examination of the kernel of the posterior distribution of X_{J+1} reveals a normal density function, conditioned on the variances and the data; from Equation (4.13),

$$f\left(X_{J+1} \mid \sigma, \{y_{ij}\}\right) \propto exp\left[- \frac{\left(X - \frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right)^2}{2\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right)} \right].$$

Thus, the posterior distribution will have its mode at, and be symmetric around, the posterior mean, which is also the median.

The values of X_{J+1} in a symmetric interval around its posterior mean may be determined by specifying the midpoint of the interval, the half-width of the interval and the number of points at which the approximation is calculated. The midpoint and half-width of the interval are determined using the mean and standard deviation of the distribution; the number of points at which the approximations are made determines the resolution of the results, and can be specified by the model user. The posterior mean of X_{J+1} is given in Equation (4.15); the posterior standard deviation of X_{J+1}

must be approximated using LaPlace's method twice, once each in the numerator and the denominator of the posterior variance, given in Equation (4.17) and repeated here.

$$V\left[X_{J+1} \mid \{y_{ij}\}\right] = \frac{\int_{\Sigma} \left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right) \cdot g\left(\sigma \mid \{y_{ij}\}\right) d\sigma}{\int_{\Sigma} g\left(\sigma \mid \{y_{ij}\}\right) d\sigma} . \quad (5.2)$$

Note that the integrals in the denominators of Equations (5.1) and (5.2) are the same; this is the integral of the kernel of the joint posterior distribution of the variances.

5.1.2 Functions to be estimated

There are three integrals to be approximated: (1) the numerator of the posterior distribution of X_{J+1}, performed once for each value of X_{J+1} in the appropriate set; (2) the denominator of the posterior variance of X_{J+1}; and (3) the numerator of the posterior variance of X_{J+1}.

Numerator of the posterior distribution From Equation (4.11), the integrand of Equation (5.1) is

$$f\left(X_{J+1}, \sigma \mid \{y_{ij}\}\right)$$

$$= C_1 \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} - \frac{\left(X - \frac{IJ\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)^2}{2\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau}\right)}\right]. \tag{5.3}$$

<u>Denominator of the posterior variance</u> The integrand of the denominator integral in Equation (5.2) is given in Equation (4.12),

$$g\left(\sigma \mid \{y_{ij}\}\right) = \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2}$$

$$\times \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2} \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{IJ\,\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]. \tag{5.4}$$

Numerator of the posterior variance The integrand of the numerator in Equation (5.2) is also determined from Equation (4.12),

$$\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau}\right) \cdot g\left(\sigma \mid \{y_{ij}\}\right)$$

$$= \left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau}\right)\left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2}$$

$$\times \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]. \tag{5.5}$$

5.1.3 General function for integrands

The three integrands listed above are special cases of the following common general function, to be integrated over the three variances.

$$\int_0^\infty \int_0^\infty \int_0^\infty g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right) d\sigma_R^2\, d\sigma_C^2\, d\sigma_E^2\ , \tag{5.6}$$

where, $g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)$

$$= \left(\sigma_E^2\right)^{-W1}\left(\sigma_E^2 + J\sigma_R^2\right)^{-W2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-W3}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-W4}$$

$$\times\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-W5} exp\left[-\frac{W6}{\sigma_E^2} - \frac{W7}{\sigma_E^2 + J\sigma_R^2}\right]$$

$$\times exp\left[-\frac{W8}{\sigma_E^2 + I\sigma_C^2} - \frac{W9}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} - \frac{W10}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right].$$

Table 5.1 presents the values for W1 to W4 and W6 to W9 which are the same for all three integrands; Table 5.2 presents the values for W5 and W10, which are different for the three integrands. The approximations based on LaPlace's method for integrals are described in terms of this general function in Sections 5.3 through 5.6.

Table 5.1 - Common Exponent Values

Exponent	Value	Exponent	Value
W1	$\frac{IJ - I - J + 2\alpha_E + 3}{2}$	W6	$\frac{SSE}{2} + \beta_E^{-1}$
W2	$\frac{I + 2\alpha_R + 1}{2}$	W7	$\frac{SSR}{2} + \beta_R^{-1}$
W3	$\frac{J + 2\alpha_C + 1}{2}$	W8	$\frac{SSC}{2} + \beta_C^{-1}$
W4	$\frac{1}{2}$	W9	$\frac{IJ\tau\left(\mu - \bar{y}_{-}\right)^2}{2\left(IJ + \tau\right)}$

Table 5.2 - Variable Exponent Values

	Value for		
		Variance	
Exponent	Posterior Distribution	Num.	Den.
W5	$\frac{1}{2}$	-1	0
W10	$\left(\frac{IJ+\tau}{2}\right)\left(X - \frac{IJ\bar{y}_{..}+\tau\mu}{IJ+\tau}\right)^2$	0	0

5.2 LaPlace's Method for Integral Approximation

5.2.1 General description

Using the notation in Bruijn (1961, Chapter 4), LaPlace's method provides approximate values for integrals of a general function of two parameters, $\phi(x, t)$, when t is large; the following form of integrand is used in the examples given in Bruijn's chapter 4:

$$\int_{-\infty}^{+\infty} exp\left[t \cdot h\left(x\right)\right] dx .$$

In general, asymptotically as $t \to \infty$, the value of the integral depends on the behavior of the integrand near its mode, say x^*. Outside of some neighborhood around the mode, the value of the integral is small as compared to the value of the integral inside the neighborhood; and inside the neighborhood, the integrand is approximated by a simpler function for which the integral can be evaluated. The simpler function used is a Taylor series expansion around the mode.

In statistical applications, as discussed in the articles by Leonard, Kass, Tierney and Kadane cited earlier, as well as Berger (1985, p. 266), the integrand is expressed as a product of a general function of the parameters and the posterior distribution of the parameters, which are the variables of integration. In the context of the integrals to be approximated in Equations (5.1) and (5.2), let $g(\sigma)$ denote a general function of the three variances denoted by σ; as before, $f(\sigma \mid \{y_{ij}\})$ denotes the joint posterior distribution of the three variances. The three integrals from section 5.1 be expressed in the general form:

$$\int_0^\infty g\left(\sigma\right) f\left(\sigma \mid \{y_{ij}\}\right) d\sigma = \int_0^\infty exp\left\{ log\left[g\left(\sigma\right) f\left(\sigma \mid \{y_{ij}\}\right)\right]\right\} d\sigma$$

Here, the function $log\left[g\left(\sigma\right) f\left(\sigma \mid \{y_{ij}\}\right)\right]$ replaces Bruijn's function $t \cdot h\left(x\right)$.

The t in Bruijn's expression of the integrand typically denotes the sample size in statistical applications. For the numerator of the posterior distribution,

$$g\left(\sigma\right)$$

$$\propto \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2} exp\left[- \frac{\left(X - \dfrac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)^2}{2\left(\dfrac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau}\right)}\right],$$

where the proportionality is due to the difference in the normalizing constants in the numerator and denominator integrands in Equation (5.1). For the denominator of the posterior variance,

$$g\left(\sigma\right) = 1 .$$

And for the numerator of the posterior variance,

$$g\left(\sigma\right) = \frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} .$$

When applying LaPlace's method to the integrals in Expression (5.6), care must be used in observing the values of the modes of the integrand, since the domains of integration for the variances are limited to (0, ∞). Depending on the sample data and prior parameter values, the modes of the integrand may occur at interior points of the domains of integration, or they may occur at the zero boundary in the row variance and/or column variance dimensions; the error variance mode will always be a positive value. The application of LaPlace's method when the mode is positive is different than the application when the mode is at the zero boundary. Thus, four different situations may arise depending upon the values of the modes for the column variance and row variance; these are summarized in the Table 5.3.

The strategy is to numerically find the values of the three variances that maximize the integrand of Expression (5.6) over the non-negative octant of the three dimension parameter space. The type of analysis is determined by the values of the row variance mode and column variance

Table 5.3 - Types of Analysis

Type	σ_C^2 mode	σ_R^2 mode	σ_E^2 mode
1	positive	positive	positive
2	zero	positive	positive
3	positive	zero	positive
4	zero	zero	positive

mode, following the criteria in Table 5.3. Three variations on the LaPlace method are used to handle the various situations encountered in Type 1 through 4; these are described in the following subsections: section 5.2.2 presents LaPlace's method as applied to a single integral with the mode at the zero boundary; section 5.2.3 presents LaPlace's method as applied to a single integral with a positive mode; and section 5.2.4 presents LaPlace's method as applied to multiple integrals with positive modes in each dimension. These applications of LaPlace's method are combined to give the functions used in the application of LaPlace's method to Expression (5.6) for situation Types 1 through 4; these functions are presented in sections 5.3 through 5.6, respectively. Only the Type 1 situation is directly analogous to the Kass, Tierney and Kadane application of LaPlace's method.

5.2.2 Approximating a single integral with mode at zero

For a single integral, when the maximum of the integrand occurs at the boundary $x = 0$, LaPlace's method gives, from Bruijn (1961, Section 4.3),

$$\int_0^\infty exp\left[t \cdot h\left(x\right)\right] dx \approx \left[-t \cdot h'\left(0\right)\right]^{-1} exp\left[t \cdot h\left(0\right)\right] \qquad \left(t \to \infty\right) .$$

This result is used in Type 2 situations for the integration with respect to σ_C^2, in Type 3 situations for the integration with respect to σ_R^2, or in Type 4 situations, sequentially, for the integrations with respect to σ_C^2 and σ_R^2. The approximate value for the integral is found analytically,

$$\int_0^\infty g(\sigma) f(\sigma \mid \{y_{ij}\})\, d\sigma = \int_0^\infty exp\left\{log\left[g(\sigma) f(\sigma \mid \{y_{ij}\})\right]\right\} d\sigma$$

$$\approx \left. \frac{exp\left\{log\left[g(\sigma) f(\sigma \mid \{y_{ij}\})\right]\right\}}{-\frac{\partial}{\partial\sigma} log\left[g(\sigma) f(\sigma \mid \{y_{ij}\})\right]} \right|_{\sigma=0}$$

$$\approx \left. \frac{g(\sigma) f(\sigma \mid \{y_{ij}\})}{-\frac{\partial}{\partial\sigma} log\left[g(\sigma) f(\sigma \mid \{y_{ij}\})\right]} \right|_{\sigma=0} . \tag{5.7}$$

5.2.3 Approximating a single integral with positive mode

For a single integral, when the maximum of the integrand occurs at an inner point of the interval, say x*, using the results from Bruijn (1961, Section 4.2),

$$\int_0^\infty exp\left[t \cdot h(x)\right] dx \approx (2\pi)^{1/2}\left[-t \cdot h''(x^*)\right]^{-1/2} exp\left[t \cdot h(x^*)\right] \quad (t \to \infty).$$

This result is only used in a Type 4 situation, for integration over σ_E^2, after having used Equation (5.7) sequentially with respect to the row and column variances. Due to the complexity of the integrand its mode cannot be found analytically. The mode of the integrand, and the approximate value of the integral using LaPlace's method, are found numerically using the computer program described in Chapter 6.

To find the mode of the integrand, the computer program uses an optimization subroutine from IMSL (1978a). This subroutine produces as output the minimum value of a multi-dimension function, as well as the points where that minimum value is attained; the subroutine uses as input separate, additional subroutines to evaluate the objective function, the gradient vector of first partial derivatives, and the Hessian matrix of second partial derivatives. To simplify the structure of the computer program, the *log* of the inverse of the integrand in Expression (5.6) is used as the objective function of the minimization subroutine. Let $log\left[g(x)\right]^{-1}$ denote the *log* of the inverse of $g(x)$; then the value of x that maximizes $g(x)$ is the same value that minimizes $log\left[g(x)^{-1}\right]$. And evaluating $g(x)$ at its mode is equivalent to evaluating $exp\left\{-log\left[g(x)^{-1}\right]\right\}$ at the same point. The reason for adopting this transformation is computational efficiency; an advantage of this transformation is being able to use the subroutine which computes the Hessian matrix for the minimization subroutine to also evaluate the denominator in the LaPlace approximation.

The approximation is expressed as:

$$\int_0^\infty g\left(\sigma_E^2\right)f\left(\sigma_E^2 \mid \{y_{ij}\}\right)d\sigma_E^2 = \int_0^\infty exp\left\{log\left[g\left(\sigma_E^2\right)f\left(\sigma_E^2 \mid \{y_{ij}\}\right)\right]\right\}d\sigma_E^2$$

$$= \int_0^\infty exp\left\{-log\left[\left\{g\left(\sigma_E^2\right)f\left(\sigma_E^2 \mid \{y_{ij}\}\right)\right\}^{-1}\right]\right\}d\sigma_E^2$$

$$\approx \left. \frac{exp\left\{-log\left[\left\{g\left(\sigma_E^2\right)f\left(\sigma_E^2 \mid \{y_{ij}\}\right)\right\}^{-1}\right]\right\}}{\left\{\frac{\partial^2}{\left(\partial\sigma_E^2\right)^2} log\left[\left\{g\left(\sigma_E^2\right)f\left(\sigma_E^2 \mid \{y_{ij}\}\right)\right\}^{-1}\right]\right\}^{1/2}} \right|_{\sigma_E^2 = \sigma_E^{2*}},$$

(5.8)

omitting constants, where $\left(\sigma_E^2\right)^*$ denotes the mode of $g\left(\sigma_E^2\right)f\left(\sigma_E^2 \mid \{y_{ij}\}\right)$.

5.2.4 Approximating multiple integrals with all modes positive

For n-tuple integrals, when the maxima of the integrands occur at inner points of the intervals, say $\mathbf{x}^*$, from Bruijn (1961, Section 4.6),

$$\int_0^\infty \cdots \int_0^\infty exp\left[t \cdot h\left(\mathbf{x}\right)\right] d\mathbf{x} \approx \left(2\pi\right)^{n/2} det\left(\mathbf{H}^*\right)^{-1/2} exp\left[t \cdot h\left(\mathbf{x}^*\right)\right]$$

$$\left(t \to \infty\right)$$

where $\mathbf{H}$ denotes the $\left(n \times n\right)$ matrix with $\left(i, j\right)$ elements

$$H_{ij} = -\left[\frac{\partial^2}{\partial x_i \partial x_j} t \cdot h\left(\mathbf{x}\right)\right]$$

and $\mathbf{H}^*$, denotes the matrix $\mathbf{H}$ evaluated at the modes, $\mathbf{x}^*$.

This result is used in Type 1, Type 2, or Type 3 situations, for the integrations over the variances with positive modes in Expression (5.6). In each of those situations, the modes cannot be found analytically, so the approximation is evaluated numerically. Here, σ denotes the set of variances with positive modes, which set changes depending on the data type: for Type 1, $\sigma = \{\sigma_E^2, \sigma_R^2, \sigma_C^2\}$ with $n = 3$; for Type 2, $\sigma = \{\sigma_E^2, \sigma_R^2\}$ with $n = 2$; and for Type 3, $\sigma = \{\sigma_E^2, \sigma_C^2\}$ with $n = 2$.

$$\int_\Sigma g\left(\sigma\right)f\left(\sigma \mid \{y_{ij}\}\right)d\sigma = \int_\Sigma exp\left\{-log\left[\left\{g\left(\sigma\right)f\left(\sigma \mid \{y_{ij}\}\right)\right\}^{-1}\right]\right\}d\sigma$$

$$\approx \left.\frac{exp\left\{-log\left[\left\{g\left(\sigma\right)f\left(\sigma \mid \{y_{ij}\}\right)\right\}^{-1}\right]\right\}}{det(\mathbf{H})^{1/2}}\right|_{\sigma=\sigma^*}, \qquad (5.9)$$

omitting constants, where σ^* denotes the modes of the variances and **H** denotes the $(n \times n)$ matrix with (i, j) elements

$$H_{ij} = \left[\frac{\partial^2 log\left[\left\{g\left(\sigma\right)f\left(\sigma \mid \{y_{ij}\}\right)\right\}^{-1}\right]}{\partial\sigma_i\,\partial\sigma_j}\right].$$

5.3 Type 1: All Modes Are Positive

When the column variance, row variance, and error variance modes all have positive values, an approximate value for Expression (5.6) is found by applying Equation (5.9) with respect to all three variances.

$$\int_0^\infty\int_0^\infty\int_0^\infty g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)d\sigma_R^2\,d\sigma_C^2\,d\sigma_E^2 \qquad (5.10)$$

$$\approx \left.\frac{exp\left\{-log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]\right\}}{det(\mathbf{H})^{1/2}}\right|_{\sigma_R^2=\sigma_R^{2*},\,\sigma_C^2=\sigma_C^{2*},\,\sigma_E^2=\sigma_E^{2*}},$$

where $\sigma_{.}^{2*}$ denotes the mode, and **H** denotes the (3×3) matrix with (i,j) elements

$$H_{ij} = \left[\frac{\partial^2 log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\partial\left(\sigma_i^2\right)\partial\left(\sigma_j^2\right)}\right], \qquad i,j \in \{R, C, E\} .$$

The approximation is performed numerically, using the following functions which are derived in Appendix K. For the numerator:

$$log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]$$

$$= W1 \cdot log\left(\sigma_E^2\right) + W2 \cdot log\left(\sigma_E^2 + J\sigma_R^2\right) + W3 \cdot log\left(\sigma_E^2 + I\sigma_C^2\right)$$

$$+ W4 \cdot log\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + W5 \cdot log\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]$$

$$+ \frac{W6}{\sigma_E^2} + \frac{W7}{\sigma_E^2 + J\sigma_R^2} + \frac{W8}{\sigma_E^2 + I\sigma_C^2} + \frac{W9}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}$$

$$+ \frac{W10}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2} .$$

For the denominator, the second derivatives of the integrand are used in the Hessian matrix.

$$\frac{\partial^2 log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\left[\partial\left(\sigma_R^2\right)\right]^2}$$

$$= -\frac{J^2 \cdot W2}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} - \frac{J^2 \cdot W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{J^2 \cdot W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \frac{2J^2 \cdot W7}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} + \frac{2J^2 \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3} + \frac{2J^2 \cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3};$$

$$\frac{\partial^2 log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\left[\partial\left(\sigma_C^2\right)\right]^2}$$

$$= -\frac{I^2 \cdot W3}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{I^2 \cdot W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{\left(I + IJ + \tau\right)^2 \cdot W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \frac{2I^2 \cdot W8}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \frac{2I^2 \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3} + \frac{2\left(I + IJ + \tau\right)^2 \cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3};$$

$$\frac{\partial^2 log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\left[\partial\left(\sigma_E^2\right)\right]^2}$$

$$= -\frac{W1}{\left(\sigma_E^2\right)^2} - \frac{W2}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} - \frac{W3}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2}$$

$$- \frac{W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} + \frac{2 \cdot W6}{\left(\sigma_E^2\right)^3} + \frac{2 \cdot W7}{\left(\sigma_E^2 + J\sigma_R^2\right)^3}$$

$$+ \frac{2 \cdot W8}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \frac{2 \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3} + \frac{2 \cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3};$$

$$\frac{\partial^2 log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\partial\left(\sigma_R^2\right)\partial\left(\sigma_C^2\right)}$$

$$= -\frac{IJ \cdot W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{J\left(I + IJ + \tau\right)W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} + \frac{2IJ \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3}$$

$$+ \frac{2J\left(I + IJ + \tau\right)W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3};$$

$$\frac{\partial^2 log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\partial\left(\sigma_R^2\right)\partial\left(\sigma_E^2\right)}$$

$$= -\frac{J \cdot W2}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} - \frac{J \cdot W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{J \cdot W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \frac{2J \cdot W7}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} + \frac{2J \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3} + \frac{2J \cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3};$$

and, $\frac{\partial^2 log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\partial\left(\sigma_C^2\right)\partial\left(\sigma_E^2\right)}$

$$= -\frac{I \cdot W3}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{I \cdot W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{\left(I + IJ + \tau\right)W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \frac{2I \cdot W8}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \frac{2I \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3} + \frac{2\left(I + IJ + \tau\right)W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3}.$$

5.4 Type 2: Column Variance Mode = 0

When the column variance mode is at the zero boundary, and the row variance and error variance modes have positive values, an approximate value for Expression (5.6) is found in a two step process. First, Equation (5.7) is applied to Expression (5.6) with respect to the column variance;

second, Equation (5.9) is applied to the result from the first step with respect to the row variance and error variance.

Step 1: Equation (5.7) is applied to Expression (5.6) with respect to the column variance; details are presented in Appendix L.1.

$$\int_0^\infty g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right) d\sigma_C^2 \approx \left. \frac{g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)}{-\dfrac{\partial}{\partial\left(\sigma_C^2\right)} \log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)\right]} \right|_{\sigma_C^2 = 0}$$

$$\approx \left(\sigma_E^2\right)^{-(W1 + W3)} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(W2 + W4 + W5)} exp\left[-\frac{W6 + W8}{\sigma_E^2} - \frac{W7 + W9 + W10}{\sigma_E^2 + J\sigma_R^2} \right]$$

$$\times \left(\begin{array}{c} \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ -\dfrac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right)^{-1} \qquad (5.11)$$

$$\approx \tilde{g}\left(\sigma_R^2, \sigma_E^2\right), \text{ say.}$$

Step 2: Equation (5.9) is applied to $\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)$.

$$\int_0^\infty \int_0^\infty \tilde{g}\left(\sigma_R^2, \sigma_E^2\right) d\sigma_R^2 \, d\sigma_E^2 \approx \left. \frac{exp\left\{-\log\left[\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)^{-1}\right]\right\}}{\det(\mathbf{H})^{1/2}} \right|_{\sigma_R^2 = \sigma_R^{2*}, \, \sigma_E^2 = \sigma_E^{2*}},$$

(5.12)

where $\sigma_{\cdot}^{2*}$ denotes the mode, and **H** denotes the (2×2) matrix with (i,j) elements

$$H_{ij} = \left[\frac{\partial^2 log\left[g\left(\sigma_R^2, \sigma_E^2\right)^{-1}\right]}{\partial\left(\sigma_i^2\right)\partial\left(\sigma_j^2\right)}\right], \qquad i,j \in \{R, E\}.$$

The approximation is performed numerically, using the following functions which are derived in Appendix L.2. For the numerator:

$$log\left[\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)^{-1}\right]$$

$$= \left(W1 + W3\right)\cdot log\left(\sigma_E^2\right) + \left(W2 + W4 + W5\right)\cdot log\left(\sigma_E^2 + J\sigma_R^2\right) + \frac{W6 + W8}{\sigma_E^2}$$

$$+ \frac{W7 + W9 + W10}{\sigma_E^2 + J\sigma_R^2} + log\left(\begin{array}{l} + \dfrac{I\cdot W3}{\sigma_E^2} + \dfrac{I\cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \dfrac{I\cdot W8}{\left(\sigma_E^2\right)^2} - \dfrac{I\cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array}\right).$$

For the denominator, the second derivatives of the integrand are used in the Hessian matrix.

$$\frac{\partial^2 log\left[\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)^{-1}\right]}{\left[\partial\left(\sigma_R^2\right)\right]^2}$$

$$= -\frac{J^2\left(W2 + W4 + W5\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2J^2\left(W7 + W9 + W10\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^3}$$

$$+ \left(\frac{2J\left[I \cdot W4 + \left(I + IJ + \tau\right)W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} - \frac{6J\left[I \cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^4}\right)$$

$$\times \left(\begin{array}{c} + \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \dfrac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array}\right)^{-1}$$

$$- \left(-\frac{J\left[I \cdot W4 + \left(I + IJ + \tau\right)W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2J\left[I \cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3}\right)^2$$

$$\times \left(\begin{array}{c} + \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \dfrac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array}\right)^{-2} ;$$

$$\frac{\partial^2 log\left[\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)^{-1}\right]}{\partial\left(\sigma_R^2\right)\partial\left(\sigma_E^2\right)}$$

$$= -\frac{J\left(W2 + W4 + W5\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2J\left(W7 + W9 + W10\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^3}$$

$$+\left(\frac{2J\left[I\cdot W4+\left(I + IJ + \tau\right)W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} - \frac{6J\left[I\cdot W9+\left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^4}\right)$$

$$\times\left(\begin{array}{c} +\dfrac{I\cdot W3}{\sigma_E^2} + \dfrac{I\cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ -\dfrac{I\cdot W8}{\left(\sigma_E^2\right)^2} - \dfrac{I\cdot W9+\left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array}\right)^{-1}$$

$$-\left(-\frac{J\left[I\cdot W4 + \left(I + IJ + \tau\right)W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2J\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3}\right)$$

$$\times\left(\begin{array}{c} -\dfrac{I\cdot W3}{\left(\sigma_E^2\right)^2} - \dfrac{I\cdot W4 + \left(I + IJ + \tau\right)W5}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \\ \\ +\dfrac{2I\cdot W8}{\left(\sigma_E^2\right)^3} + \dfrac{2\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} \end{array}\right)$$

$$\times \left(\begin{array}{l} + \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \dfrac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right)^{-2} ;$$

and, $\dfrac{\partial^2 log\left[\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)^{-1}\right]}{\left[\partial\left(\sigma_E^2\right)\right]^2}$

$$= -\frac{W1 + W3}{\left(\sigma_E^2\right)^2} - \frac{W2 + W4 + W5}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2\left(W6 + W8\right)}{\left(\sigma_E^2\right)^3} + \frac{2\left(W7 + W9 + W10\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^3}$$

$$+ \left(\begin{array}{l} + \dfrac{2I \cdot W3}{\left(\sigma_E^2\right)^3} + \dfrac{2\left[I \cdot W4 + \left(I + IJ + \tau\right)W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} \\ \\ - \dfrac{6I \cdot W8}{\left(\sigma_E^2\right)^4} - \dfrac{6\left[I \cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^4} \end{array} \right)$$

$$\times \left(\begin{array}{l} + \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \dfrac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right)^{-1}$$

$$
-\left(\begin{array}{l} -\dfrac{I\cdot W3}{\left(\sigma_E^2\right)^2} - \dfrac{I\cdot W4 + \left(I + IJ + \tau\right)W5}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \\ \\ + \dfrac{2I\cdot W8}{\left(\sigma_E^2\right)^3} + \dfrac{2\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} \end{array} \right)^2
$$

$$
\times \left(\begin{array}{l} +\dfrac{I\cdot W3}{\sigma_E^2} + \dfrac{I\cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ -\dfrac{I\cdot W8}{\left(\sigma_E^2\right)^2} - \dfrac{I\cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right)^{-2} .
$$

5.5 Type 3: Row Variance Mode = 0

When the row variance mode is at the zero boundary, and the row variance and error variance modes have positive values, an approximate value for Expression (5.6) is found in a two step process. First, Equation (5.7) is applied to Expression (5.6) with respect to the row variance; second, Equation (5.9) is applied to the result from the first step with respect to the column variance and error variances.

Step 1: Equation (5.7) is applied to Expression (5.6) with respect to the row variance; details are presented in Appendix M.1.

$$
\int_0^\infty g\left(\sigma_C^2, \sigma_R^2, \sigma_E^2\right) d\sigma_R^2 \approx \left. \frac{g\left(\sigma_C^2, \sigma_R^2, \sigma_E^2\right)}{-\dfrac{\partial}{\partial\left(\sigma_R^2\right)} \log\left[g\left(\sigma_C^2, \sigma_R^2, \sigma_E^2\right)\right]} \right|_{\sigma_R^2=0}
$$

$$\approx \left(\sigma_E^2\right)^{-(W1+W2)} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(W3+W4)} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-W5}$$

$$\times exp\left[-\frac{W6+W7}{\sigma_E^2} - \frac{W8+W9}{\sigma_E^2 + I\sigma_C^2} - \frac{W10}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right]$$

$$\times \left(\begin{array}{l} +\dfrac{J \cdot W2}{\sigma_E^2} + \dfrac{J \cdot W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{J \cdot W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ \\ -\dfrac{J \cdot W7}{\left(\sigma_E^2\right)^2} - \dfrac{J \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{J \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array}\right)^{-1} \tag{5.13}$$

$$\approx \tilde{g}\left(\sigma_C^2, \sigma_E^2\right), \text{ say.}$$

Step 2: Equation (5.9) is applied to $\tilde{g}\left(\sigma_C^2, \sigma_E^2\right)$.

$$\int_0^\infty \int_0^\infty \tilde{g}\left(\sigma_C^2, \sigma_E^2\right) d\sigma_C^2 \, d\sigma_E^2 \approx \left. \frac{-exp\left\{log\left[\tilde{g}\left(\sigma_C^2, \sigma_E^2\right)^{-1}\right]\right\}}{\det(\mathbf{H})^{1/2}} \right|_{\sigma_C^2 = \sigma_C^{2*}, \sigma_E^2 = \sigma_E^{2*}}, \tag{5.14}$$

where $\sigma_.^{2*}$ denotes the mode, and **H** denotes the (2×2) matrix with (i, j) elements

$$H_{ij} = \left[\frac{\partial^2 log\left[\tilde{g}\left(\sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\partial\left(\sigma_i^2\right)\partial\left(\sigma_j^2\right)}\right], \qquad i,j \in \{C, E\}.$$

The approximation is performed numerically, using the following functions which are derived in Appendix M.2. For the numerator:

$$log\left[\tilde{g}\left(\sigma_C^2, \sigma_E^2\right)^{-1}\right]$$

$$= \left(W1 + W2\right) log\left(\sigma_E^2\right) + \left(W3 + W4\right) log\left(\sigma_E^2 + I\sigma_C^2\right)$$

$$+ W5 \cdot log\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right] + \frac{W6 + W7}{\sigma_E^2} + \frac{W8 + W9}{\sigma_E^2 + I\sigma_C^2}$$

$$+ \frac{W10}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2}$$

$$+ log\left(\begin{array}{c} + \dfrac{J \cdot W2}{\sigma_E^2} + \dfrac{J \cdot W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{J \cdot W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \dfrac{J \cdot W7}{\left(\sigma_E^2\right)^2} - \dfrac{J \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{J \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right).$$

For the denominator, the second derivatives of the integrand are used in the Hessian matrix.

$$\frac{\partial^2 log\left[\tilde{g}\left(\sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\left[\partial\left(\sigma_C^2\right)\right]^2}$$

$$= -\frac{I^2\cdot\left(W3 + W4\right)}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{\left(I + IJ + \tau\right)^2\cdot W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} + \frac{2I^2\cdot\left(W8 + W9\right)}{\left(\sigma_E^2 + I\sigma_C^2\right)^3}$$

$$+ \frac{2\left(I + IJ + \tau\right)^2\cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3}$$

$$+ \left(\begin{array}{c} + \dfrac{2I^2\cdot W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \dfrac{2\left(I + IJ + \tau\right)^2\cdot W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} \\ - \dfrac{6I^2\cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^4} - \dfrac{6\left(I + IJ + \tau\right)^2\cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^4} \end{array}\right)$$

$$\times \left(\begin{array}{c} + \dfrac{W2}{\sigma_E^2} + \dfrac{W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \dfrac{W7}{\left(\sigma_E^2\right)^2} - \dfrac{W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array}\right)^{-1}$$

$$
- \left(\begin{array}{l} - \dfrac{I \cdot W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{\left(I + IJ + \tau\right) W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \\[2em] + \dfrac{2I \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \dfrac{2\left(I + IJ + \tau\right) W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} \end{array} \right)^2
$$

$$
\times \left(\begin{array}{l} + \dfrac{W2}{\sigma_E^2} + \dfrac{W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\[2em] - \dfrac{W7}{\left(\sigma_E^2\right)^2} - \dfrac{W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right)^{-2} ;
$$

$$\frac{\partial^2 log\left[\tilde{g}\left(\sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\partial\left(\sigma_C^2\right)\partial\left(\sigma_E^2\right)}$$

$$= -\frac{I\cdot\left(W3 + W4\right)}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{\left(I + IJ + \tau\right)\cdot W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} + \frac{2I\cdot\left(W8 + W9\right)}{\left(\sigma_E^2 + I\sigma_C^2\right)^3}$$

$$+ \frac{2\left(I + IJ + \tau\right)\cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3}$$

$$+ \left(\begin{array}{c} + \dfrac{2I\cdot W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \dfrac{2\left(I + IJ + \tau\right)\cdot W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} \\ - \dfrac{6I\cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^4} - \dfrac{6\left(I + IJ + \tau\right)\cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^4} \end{array} \right)$$

$$\times \left(\begin{array}{c} + \dfrac{W2}{\sigma_E^2} + \dfrac{W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \dfrac{W7}{\left(\sigma_E^2\right)^2} - \dfrac{W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right)^{-1}$$

$$
-\left(\begin{aligned}&-\frac{W2}{\left(\sigma_E^2\right)^2}-\frac{W4}{\left(\sigma_E^2+I\sigma_C^2\right)^2}-\frac{W5}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^2}\\&+\frac{2\cdot W7}{\left(\sigma_E^2\right)^3}+\frac{2\cdot W9}{\left(\sigma_E^2+I\sigma_C^2\right)^3}+\frac{2\cdot W10}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^3}\end{aligned}\right)
$$

$$
\times\left(\begin{aligned}&-\frac{I\cdot W4}{\left(\sigma_E^2+I\sigma_C^2\right)^2}-\frac{\left(I+IJ+\tau\right)W5}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^2}\\&+\frac{2I\cdot W9}{\left(\sigma_E^2+I\sigma_C^2\right)^3}+\frac{2\left(I+IJ+\tau\right)W10}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^3}\end{aligned}\right)
$$

$$
\times\left(\begin{aligned}&+\frac{W2}{\sigma_E^2}+\frac{W4}{\sigma_E^2+I\sigma_C^2}+\frac{W5}{\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2}\\&-\frac{W7}{\left(\sigma_E^2\right)^2}-\frac{W9}{\left(\sigma_E^2+I\sigma_C^2\right)^2}-\frac{W10}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^2}\end{aligned}\right)^{-2};
$$

and, $\dfrac{\partial^2 log\left[\tilde{g}\left(\sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\left[\partial\left(\sigma_E^2\right)\right]^2}$

$$= -\frac{W1 + W2}{\left(\sigma_E^2\right)^2} - \frac{W3 + W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} + \frac{2\left(W6 + W7\right)}{\left(\sigma_E^2\right)^3}$$

$$+ \frac{2\left(W8 + W9\right)}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \frac{2 \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3}$$

$$+ \left(\begin{array}{c} + \dfrac{2 \cdot W2}{\left(\sigma_E^2\right)^3} + \dfrac{2 \cdot W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \dfrac{2 \cdot W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} \\ - \dfrac{6 \cdot W7}{\left(\sigma_E^2\right)^4} - \dfrac{6 \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^4} - \dfrac{6 \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^4} \end{array} \right)$$

$$\times \left(\begin{array}{c} + \dfrac{W2}{\sigma_E^2} + \dfrac{W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \dfrac{W7}{\left(\sigma_E^2\right)^2} - \dfrac{W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right)^{-1}$$

$$- \left(\begin{array}{l} - \dfrac{W2}{\left(\sigma_E^2\right)^2} - \dfrac{W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \\ + \dfrac{2 \cdot W7}{\left(\sigma_E^2\right)^3} + \dfrac{2 \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \dfrac{2 \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} \end{array} \right)^2$$

$$\times \left(\begin{array}{l} + \dfrac{W2}{\sigma_E^2} + \dfrac{W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \dfrac{W7}{\left(\sigma_E^2\right)^2} - \dfrac{W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right)^{-2} .$$

5.6 Type 4: Column Variance Mode = Row Variance Mode = 0

When the column variance mode and the row variance mode are each at the zero boundary and the error variance mode has positive value, an approximate value for Expression (5.6) is found in a three step process. First, Equation (5.7) is applied to Expression (5.6) with respect to the column variance; second, Equation (5.7) is applied to the function from step 1 with respect to the row variance; and, third, Equation (5.8) is applied to the function from step 2 with respect to the error variance.

Step 1: This is the same analysis as performed in step 1 in Section 5.4, with the details presented in Appendix L.1.

$$\int_0^\infty g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right) d\sigma_C^2 \approx \left. \frac{g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)}{-\dfrac{\partial}{\partial\left(\sigma_C^2\right)} log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)\right]} \right|_{\sigma_C^2 = 0}$$

$$\approx \left(\sigma_E^2\right)^{-(W1 + W3)} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(W2 + W4 + W5)} exp\left[-\frac{W6 + W8}{\sigma_E^2} - \frac{W7 + W9 + W10}{\sigma_E^2 + J\sigma_R^2}\right]$$

$$\times \left(\frac{\dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2}}{-\dfrac{I \cdot W\,8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2}} \right)^{-1}$$

$$\approx \tilde{g}\left(\sigma_R^2, \sigma_E^2\right).$$

Step 2: Equation (5.7) is applied to $\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)$ with respect to the row variance; details are presented in Appendix N.1.

$$\int_0^\infty \tilde{g}\left(\sigma_R^2, \sigma_E^2\right) d\sigma_R^2 \approx \left. \frac{\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)}{-\dfrac{\partial}{\partial\left(\sigma_R^2\right)} log\left[\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)\right]} \right|_{\sigma_R^2 = 0}$$

$$\approx \left(\sigma_E^2\right)^{-(W1 + W2 + W3 + W4 + W5 - 4)} exp\left[- \frac{W6 + W7 + W8 + W9 + W10}{\sigma_E^2} \right]$$

$$\times \begin{pmatrix} + J(W2+W4+W5)\left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J\left[I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J(W2+W4+W5)\left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ - J(W7+W9+W10)\left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ + 2J\left[I \cdot W9 + \left(I + IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ + J(W7+W9+W10)\left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{pmatrix}^{-1}$$

(5.15)

$$\approx \tilde{\tilde{g}}\left(\sigma_E^2\right).$$

Step 3: Equation (5.8) is applied to $\tilde{\tilde{g}}\left(\sigma_E^2\right)$.

$$\int_0^\infty \tilde{\tilde{g}}\left(\sigma_E^2\right) d\sigma_E^2 \approx \left. \frac{exp\left\{ -log\left[\tilde{\tilde{g}}\left(\sigma_E^2\right)^{-1} \right] \right\}}{\left\{ \frac{\partial^2}{\left[\partial\left(\sigma_E^2\right)\right]^2} log\left[\tilde{\tilde{g}}\left(\sigma_E^2\right)^{-1} \right] \right\}^{1/2}} \right|_{\sigma_E^2 = \sigma_E^{2*}} , \qquad (5.16)$$

where $\sigma_E^{2\,*}$ denotes the mode of the error variance. The approximation is performed numerically, using the following functions which are derived in Appendix N.2. For the numerator:

$$log\left[\bar{\bar{g}}\left(\sigma_E^2\right)^{-1}\right]$$

$$= \left(W1 + W2 + W3 + W4 + W5 - 4\right) log\left(\sigma_E^2\right) + \frac{W6 + W7 + W8 + W9 + W10}{\sigma_E^2}$$

$$+ log\begin{pmatrix} + J(W2{+}W4{+}W5)\left[I\cdot W3 + I\cdot W4 + \left(I{+}IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J\left[I\cdot W4 + \left(I{+}IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J(W2{+}W4{+}W5)\left[I\cdot W8 + I\cdot W9 + \left(I{+}IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ - J(W7{+}W9{+}W10)\left[I\cdot W3 + I\cdot W4 + \left(I{+}IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ + 2J\left[I\cdot W9 + \left(I{+}IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ + J(W7{+}W9{+}W10)\left[I\cdot W8 + I\cdot W9 + \left(I{+}IJ + \tau\right)W10\right] \end{pmatrix};$$

The denominator uses the second derivative of the integrand.

$$\frac{\partial^2 log\left[\tilde{\tilde{g}}\left(\sigma_E^2\right)^{-1}\right]}{\left[\partial\left(\sigma_E^2\right)\right]^2}$$

$$= -\frac{W1 + W2 + W3 + W4 + W5 - 4}{\left(\sigma_E^2\right)^2} + \frac{2\left(W6 + W7 + W8 + W9 + W10\right)}{\left(\sigma_E^2\right)^3}$$

$$+\begin{pmatrix} +2\left(W2 + W4 + W5\right)\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right] \\ -2\left[I\cdot W4 + \left(I + IJ + \tau\right)W5\right] \end{pmatrix}$$

$$\times\begin{pmatrix} +\left(W2+W4+W5\right)\left[I\cdot W3+I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ -\left[I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ -\left(W2+W4+W5\right)\left[I\cdot W8+I\cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right) \\ -\left(W7+W9+W10\right)\left[I\cdot W3+I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right) \\ +2\left[I\cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right) \\ +\left(W7+W9+W10\right)\left[I\cdot W8+I\cdot W9+\left(I+IJ+\tau\right)W10\right] \end{pmatrix}^{-1}$$

$$
- \begin{pmatrix}
+ 2\,(W2+W4+W5)\left[I\cdot W3+I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right) \\
- 2\left[I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right) \\
- (W2+W4+W5)\left[I\cdot W8+I\cdot W9+\left(I+IJ+\tau\right)W10\right] \\
- (W7+W9+W10)\left[I\cdot W3+I\cdot W4+\left(I+IJ+\tau\right)W5\right] \\
+ 2\left[I\cdot W9+\left(I+IJ+\tau\right)W10\right]
\end{pmatrix}^2
$$

$$
\times \begin{pmatrix}
+ (W2+W4+W5)\left[I\cdot W3+I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right)^2 \\
- \left[I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right)^2 \\
- (W2+W4+W5)\left[I\cdot W8+I\cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right) \\
- (W7+W9+W10)\left[I\cdot W3+I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right) \\
+ 2\left[I\cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right) \\
+ (W7+W9+W10)\left[I\cdot W8+I\cdot W9+\left(I+IJ+\tau\right)W10\right]
\end{pmatrix}^{-2} .
$$

CHAPTER 6

DEMONSTRATION OF THE APPROXIMATION METHODOLOGY

This chapter describes and demonstrates the computer program written to implement the approximation methodology developed in Chapter 5. Examples using non-informative and informative prior distributions are given for each of the four types of data sets encountered using the two-way random effects model. Comparisons between the Bayesian and sampling theory results are made for the expected value, standard deviation, and selected probability intervals of the mean of the $(J+1)^{th}$ replication of the microsimulation model. The computer program is described in Section 6.1. In Section 6.2 the demonstration uses the data set generated by the Nakamura simulation model with 1000 replications; this is a Type 1 data set. In Section 6.3 the demonstration uses a data set taken from the first 400 replications of the Nakamura simulation model to demonstrate a Type 2 data set. In Section 6.4 the demonstration uses the transpose of the Type 2 data set as a Type 3 data set. In Section 6.5, the demonstration uses a generated Type 4 data set. Some general comments about the application of the LaPlace method to various data sets are made in Section 6.5.

6.1 The Approximation Computer Program

A computer program written in the FORTRAN language calculates the posterior distribution of the mean of the $(J+1)^{th}$ replication of a microsimulation model, based upon the analysis developed in Chapter 5.

The computer program code is given in Appendix O. An outline of the analysis program is presented in Table 6.1. For comparison purposes, the program also calculates confidence intervals using sampling theory results developed in Chapter 3. Comparisons are made between the results

Table 6.1 - Approximation Program Outline

Step	Operation
1	enter prior distribution parameters
2	enter sample data
3	calculate variances using sampling theory
	FOR POSTERIOR STANDARD DEVIATION
	FOR NUMERATOR
4	find mode
5	determine type
6	calculate estimated value
	FOR DENOMINATOR
7	find mode
8	determine type
9	calculate estimated value
10	calculate estimated standard deviation
	FOR POSTERIOR DISTRIBUTION
	AT POSTERIOR MEAN
11	find mode
12	determine type
13	calculate estimated value
	LOOP THROUGH UPPER HALF OF DISTRIBUTION
14	calculate deviation from mean
15	find mode
16	determine type
17	calculate estimated value
18	assign same value to symmetric deviation below mean
	END LOOP
19	normalize to proper probability distribution
20	find selected Bayesian HPD credible sets

from the Bayesian and sampling theory approaches numerically and graphically by calculating the corresponding expected values, standard deviations, and selected probability intervals, as well as by displaying the analogous distributions.

In step 1 of the program, the user is queried for the values of the parameters of the prior distributions. This operation is performed in the first part of subroutine INPUTS. The user has the option of specifying non-informative priors, or specifying a numerical value for a parameter if using informative priors, for any of the parameters. The variable GAMMA(i) is used as the inverse of the beta parameter for each of the three variances; for non-informative prior on a variance, GAMMA(i) is set equal to zero, since the non-informative value of the beta parameter is infinity.

In step 2, the sample data from the microsimulation experiment is entered; this operation is performed in the second part of subroutine INPUTS. The user has the option of directly entering the values of the sample sufficient statistics in the form $\left\{I, J, \bar{y}_{..}, SSR, SSC, SSE\right\}$, or having the values of the sample sufficient statistics read from a file in the form

$$\left\{I, J, \left(\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij}\right), \left(\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij}^{2}\right), \sum_{i=1}^{I}\left(\sum_{j=1}^{J} y_{ij}\right)^{2}, \sum_{j=1}^{J}\left(\sum_{i=1}^{I} y_{ij}\right)^{2}\right\}.$$

Appendix P contains a listing of the FORTRAN program which calculates this set of sufficient statistics from an I×J matrix of values from the experiment. After the entry of the sample data, the values of the common exponents, given in Table 5.1, are calculated.

In step 3, the estimates of the variances using the method-of-moments are calculated, following Equations (3.8) to (3.10). This operation is performed in subroutine SMPDAT. If the row or column variance estimate has a negative value, that variance estimate is set equal to 0. The method-of-moments expected value of the mean of the $(J+1)^{th}$ replication is equal to the sample average, $\bar{y}_{..}$. The standard deviation of the mean of the $(J+1)^{th}$ replication is calculated as the square root of the variance given in Equation (3.7), which is equivalent to using the variance from either Equations (3.11) to (3.14) depending on the negativity of the row and/or column variance estimates.

The posterior standard deviation of the mean of the $(J+1)^{th}$ replication is calculated in steps 4 through 10. This operation is performed in subroutine MOMNTS. The numerator of the posterior variance is calculated in steps 4 through 6; and the denominator of the posterior variance is calculated in steps 7 through 9. The standard deviation is calculated in step 10. The procedure for calculating the numerator of the posterior variance is the first application of the LaPlace method; similar procedures are used in steps 7 through 8 for the denominator of the posterior variance and in steps 11 through 13 and 15 through 17 for the posterior distribution, with minor modifications accomplished by changing the exponent values as given in Table 5.2. The procedure for steps 4 through 7 is explained in detail; while the detail is not given for the procedures for the similar, subsequent steps.

In step 4, subroutine DBCOAH, from IMSL, Inc. (1987a), is used to find the mode of the numerator of the posterior variance, given in Equation (5.5), omitting the constant term $(IJ + \tau)^{-1}$ which will be included in step 10. The right side of Equation (5.5) is expressed as the general function of integrands given in Expression (5.6) using the appropriate exponent values as given in Tables 5.1 and 5.2. Subroutine DBCOAH is restricted to optimize over the nonnegative portion of $\Re^n$. Since subroutine DBCOAH finds the value that minimizes an n-dimensional user-supplied objective function, the log of the inverse of Expression (5.6), $log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]$, is used as the objective function for subroutine DBCOAH; this objective function is written in subroutine LFNC1, with $n = 3$. In addition to the objective function, subroutine DBCOAH also requires user-supplied subroutines which contain the gradient vector of first derivatives of the objective function and the HESSIAN matrix of second derivatives of the objective function; these are written in subroutines GRAD1 and HESS1. The method-of-moments estimates for the variances, with a value of zero used for each variance estimate that is negative, are used as the starting points of the search in subroutine DBCOAH. Output from subroutine DBCOAH consists of the minimum value of the objective function, and the n-tuple set of points which minimizes the objective function. These points are the maximum likelihood estimates of the variances for the function given in Equation (5.5).

In step 5, the data set type is determined based upon the values of the variance estimates and the criteria given in Table 5.3. The output from subroutine DBCOAH provides the values of the row, column and error variances which optimize the objective function, defined over the entire $\Re^3$ space; these values are the numerical estimates of the posterior modes for the respective variances. If the row and column variances are positive, the data set is Type 1; if the row variance is positive and the column variance is negative, the data set is Type 2; if the row variance is negative and the column variance is positive, the data set is Type 3; and if the row and column variances are negative, the data set is Type 4. If the data set is Type 1, program control remains in subroutine MOMNTS for step 6; if the data set is not Type 1, program control passes to subroutine ESTIM2, ESTIM3 or ESTIM4 for step 6 depending on the data set type. The positive estimate(s) of the variance(s) are passed to the respective ESTIM*i* subroutine for use as the starting point(s) of the search.

In step 6, if the data set is Type 1, the minimum value of the objective function is currently available as one of the outputs from subroutine DBCOAH; subroutine HESS1 calculates the Hessian matrix. These values are used to calculate the log of the right side of Equation (5.10).

If the data set is Type 2, subroutine ESTIM2 performs a procedure similar to step 4. Subroutine DBCOAH is used with $n = 2$ to optimize over the row and error variances using subroutines LFNC2, GRAD2, and HESS2 which reflect the objective function given in Equation (5.11), $log\left[\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)^{-1}\right]$, and an output value is the minimum value of the objective function. Subroutine HESS2 calculates the Hessian matrix. These values are used to calculate the log of the right side of Equation (5.12).

If the data set is Type 3, subroutine ESTIM3 performs a procedure similar to step 4. Subroutine DBCOAH is used with $n = 2$ to optimize over the column and error variances using subroutines LFNC3, GRAD3, and HESS3 which reflect the objective function given in Equation (5.13), $log\left[\tilde{g}\left(\sigma_C^2, \sigma_E^2\right)^{-1}\right]$, and an output value is the minimum value of the objective function. Subroutine HESS3 calculates the Hessian matrix. These values are used to calculate the log of the right side of Equation (5.14).

If the data set is Type 4, subroutine ESTIM4 performs a procedure similar to step 4. Subroutine DBCOAH is used with $n = 1$ to optimize over the error variance using subroutines LFNC4, GRAD4, and HESS4 which reflect the objective function given in Equation (5.15), $log\left[\approx{g}\left(\sigma_E^2\right)^{-1}\right]$, and an output value is the minimum value of the objective function. Subroutine HESS4 calculates the second derivative. These values are used to calculate the log of the right side of Equation (5.16).

The procedure in steps 7 through 9 to calculate the denominator of the variance is performed similarly to steps 4 through 6 with the appropriate changes in the values of the exponents as given in Table 5.2.

In step 10, the posterior standard deviation is calculated from the function of the numerator of the variance calculated in step 6, the function of the denominator of the variance calculated in step 9 and the constant term from the numerator of the variance $(IJ + \tau)^{-1}$.

A discrete set of values which approximate the posterior distribution of the mean of the $(J + 1)^{th}$ replication is calculated in steps 11 through 19. This operation is performed in subroutine ESTMAT. Steps 11 through 13 calculate the value of the posterior density function at the posterior mean, using the procedure from steps 4 through 7 with the appropriate changes in the values of the exponents as given in Table 5.2. The value of W10 equals zero when X equals the posterior mean. The method-of-moments estimates for the variances, with a value of zero used for each variance estimate that is negative, are used as the starting points for the search. In order to avoid underflow errors which may occur when calculating the posterior density at points in the tail of the distribution, that is, at points that are a large number of standard deviations from the mean, the value of the posterior distribution at the mean is scaled to a value equal to 1 and the posterior distribution at points away from the mean are multiplied by the same scaling constant. The program variable CHUNK is used as the scaling constant. CHUNK is set equal to the log of the LaPlace approximation calculated in step 13, which is the log of the right side of Equation (5.10), (5.12), (5.14), or (5.16) depending on the data set type.

The posterior density is calculated at other points on the axis in a program loop for steps 14 through 18. In general, the model user may specify the set of points at which the posterior distribution is estimated by specifying the number of such points and the interval which contains them. In this program, the loop is repeated 100 times, with the posterior density being calculated for evenly spaced points on the axis starting at the mean and up to five standard deviations above the mean. In step 14, the squared term of W10 is set equal to the square of the loop index number of posterior standard deviations, and the point on the axis is calculated as the loop index number of posterior standard deviations above the mean. In steps 15 through 17 the value of the posterior distribution is calculated (similar to the procedure in steps 4 through 7), with the scaling of the result by the variable CHUNK included; the starting points for the search performed by subroutine DBCOAH on each pass through the loop are the output values which optimized the objective function from the previous pass through the loop . In step 18 the posterior density value from step 17 is assigned to the point on the axis symmetrically below the posterior mean.

The result from steps 11 through 18 is a discrete set of 201 pairs of values for the posterior density and the corresponding point on the axis. In step 19 the values of the posterior density are rescaled so that their sum equals 1; this operation is performed in subroutine NRMLIZ. There are three reasons why rescaling is necessary: (1) the use of a discrete set of points to approximate a continuous density, a problem that is not unique to the LaPlace approximation methodology; (2) the omission of the denominator of Equation (5.1) from the calculation of the posterior density function value; and, (3) the use of the program variable CHUNK to scale values to avoid underflow errors.

In step 20, the lower and upper endpoints for five selected Bayesian HPD credible sets are calculated; this operation is performed in subroutine PRCNTL. The endpoints are determined by calculating the cumulative mass function for each point on the axis and matching the appropriate percentile values for the interval endpoints with the corresponding axis values.

The program subroutines dealing with the calculations for the sampling theory distribution and intervals are not included in Table 6.1. In Section 3.3, the sampling distribution is shown to be based on the Student's-*t* distribution with degrees of freedom equal to (I-1)(J-1). For convenience, since the examples used here have large degrees of freedom, the standard normal distribution is used instead of the Student's-*t* distribution. The sampling theory intervals are calculated using the sampling theory mean and standard deviation and the appropriate percentile values from the standard normal table. The sampling theory distribution is approximated by calculating the normal probability density at an appropriate set of values on the axis, and then normalizing to a proper discrete probability mass function.

A few words of caution about the presentation of the sampling theory results are in order. The sampling theory intervals and distributions are presented in these forms so that they are analogous in form to the Bayesian theory results for direct comparisons, although they are not analogous in interpretation. The sampling theory intervals are the traditional confidence intervals, with their interpretation based upon the long-run relative frequency of random sample intervals which include the unknown, but fixed, parameter value. The sampling theory distributions are not true probability distributions, but represent a graphic depiction of the set of all sampling confidence intervals.

6.2 Nakamura Model Data Set, 1000 Replications: Type 1 Example

6.2.1 Data set description

In this section, the methodology is applied to the output data from the Nakamura simulation model of the labor force participation of married women. The output from the model for each replication is a vector consisting of the annual income for each wife. There are 1124 wives in the model; the model is replicated 1000 times. The mean annual income of all wives for an unobserved replication of the model is the variable of interest. The sample descriptive statistics are displayed in Table 6.2; the method-of-moments estimates for the variances are displayed in Table 6.3. This is a Type 1 situation, the modes of all three variance components have positive values.

Table 6.2 - Sample Descriptive Statistics

Statistic	Value	Statistic	Value
I	1124	MSR	31361945751.28
J	1000	MSC	252962953.84
$\bar{y}_{..}$	4694.98	MSE	246991805.41

Table 6.3
Method-of-Moments
Estimates of Variances

Variance	Value
Row	31114953.95
Column	5312.41
Error	246991805.41

6.2.2 Using non-informative Priors

The non-informative prior parameters for the two-way random effects model are displayed in Table 6.4; these values are also used in the non-informative priors analysis in sections 6.3.2, 6.4.2, and 6.5.2.

Table 6.4 - Prior Distribution Parameter Values

Parameter	Value	Parameter	Value
μ	0.00	α_C	0.00
τ	0.00	β_C	∞
α_R	0.00	α_E	0.00
β_R	∞	β_E	∞

Table 6.5 displays the means and standard deviations from the Bayesian and sampling theory/frequentist analyses. Table 6.6 displays the comparable intervals. Figure 6.1 displays the graphs of the comparable distributions.

The values of the means are the same for the different analyses. The Bayesian standard deviation is smaller than the frequentist standard deviation, resulting in narrower intervals.

Table 6.5 - Comparable Descriptive Statistics

	Bayesian	Frequentist
mean	4694.98	4694.98
std.dev.	175.46	182.26

Table 6.6 - Comparable Intervals

Bayesian			Frequentist	
Lower	Upper	(1-α)	Lower	Upper
4579.13	4810.83	50%	4572.14	4817.83
4490.02	4899.95	75%	4485.38	4904.59
4409.81	4980.15	90%	4395.16	4994.81
4347.43	5042.53	95%	4337.75	5052.22
4240.49	5149.47	99%	4225.47	5164.49

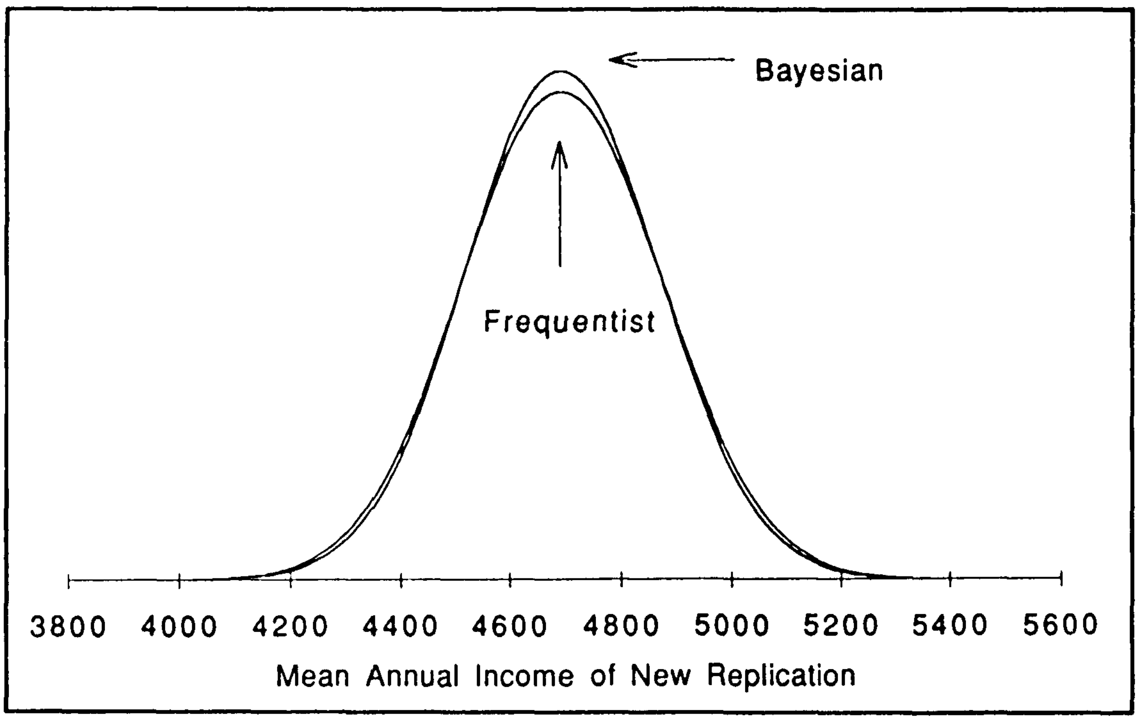

Figure 6.1 - Comparable Distributions

6.2.3 Using informative priors

The prior parameters for the informative analysis are displayed in Table 6.7. These values of the prior parameters are used to emphasize the difference between the resulting distributions.

Table 6.7 - Prior Distribution Parameter Values

Parameter	Value	Parameter	Value
μ	4000.00	α_C	4.00
τ	100000.00	β_C	0.0008
α_R	12.00	α_E	2.50
β_R	0.0001	β_E	0.005

The sample descriptive statistics and method-of-moments estimates of variances are the same as for the non-informative analysis, as displayed in Tables 6.2 and 6.3. Table 6.8 displays the comparable means and standard deviations; and Table 6.9 displays the comparable intervals. Figure 6.2 displays the graphs of the comparable distributions.

The sampling theory results are the same here as for the non-informative analysis of the previous section. The Bayesian mean is a weighted average of the sample mean and prior mean; the Bayesian standard deviation is less than the frequentist standard deviation, and less than the standard deviation from the non-informative analysis.

Table 6.8 - Comparable Descriptive Statistics

	Bayesian	Frequentist
mean	4638.20	4694.98
std.dev.	168.49	182.26

Table 6.9 - Comparable Intervals

Bayesian			Frequentist	
Lower	Upper	(1-α)	Lower	Upper
4526.78	4749.63	50%	4572.14	4817.83
4441.07	4835.34	75%	4485.38	4904.59
4363.93	4912.48	90%	4395.16	4994.81
4303.93	4972.48	95%	4337.75	5052.22
4201.07	5075.33	99%	4225.47	5164.49

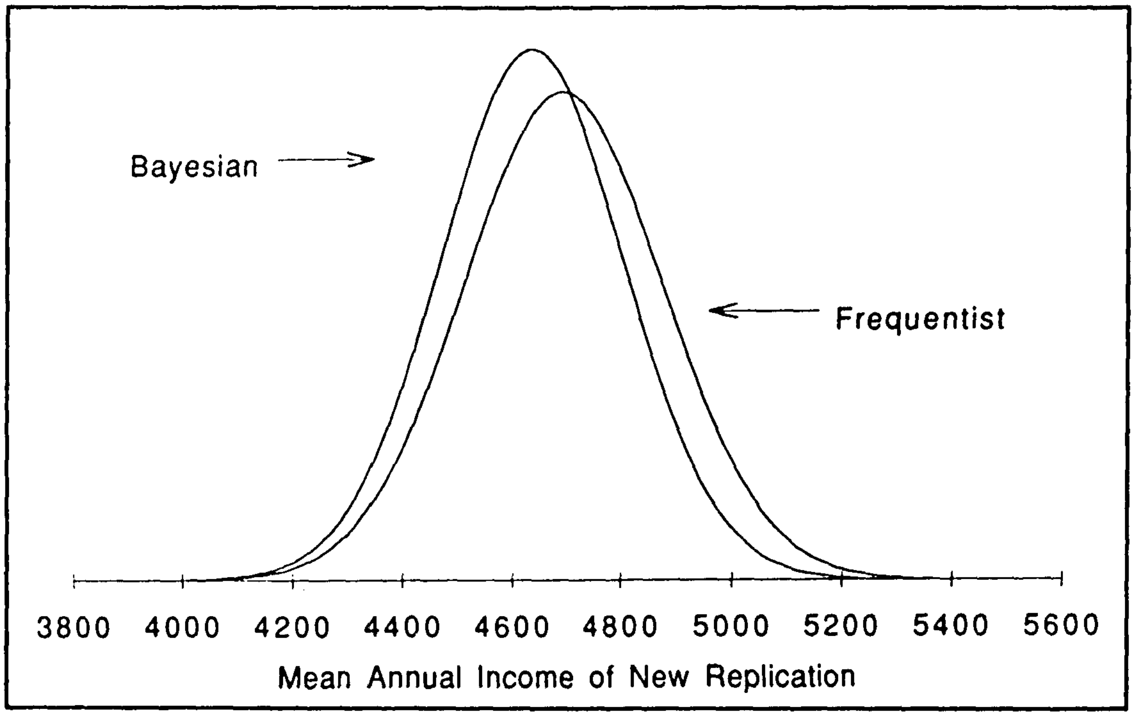

Figure 6.2 - Comparable Distributions

6.3 Nakamura Model Data Set, 400 Replications: Type 2 Example

6.3.1 Data set description

In this section, the methodology is applied to the first 400 replications of the Nakamura model. This data set is used to demonstrate the methodology for a Type 2 situation, where the mode of the column variance occurs at the zero boundary. The sample descriptive statistics are displayed in Table 6.10. The method-of-moments estimates for the variances are displayed in Table 6.11; note the negative value for the column variance estimate.

Table 6.10 - Sample Descriptive Statistics

Statistic	Value	Statistic	Value
I	1124	MSR	12347959155.68
J	400	MSC	226359015.06
$\bar{y}_{..}$	4662.97	MSE	231644954.28

Table 6.11
Method-of-Moments
Estimates of Variances

Variance	Value
Row	30290785.50
Column	- 4702.79
Error	231644954.28

6.3.2 Using non-informative Priors

The prior parameters for this analysis are displayed in Table 6.4. Table 6.12 displays the comparable means and standard deviations; and Table 6.13 displays the comparable intervals. Figure 6.3 shows the graphs of the comparable distributions.

The means have the same values. The Bayesian standard deviation is smaller than the frequentist standard deviation, resulting in narrower intervals.

Table 6.12 - Comparable Descriptive Statistics

	Bayesian	Frequentist
mean	4662.97	4662.97
std.dev.	104.34	165.72

Table 6.13 - Comparable Intervals

Bayesian			Frequentist	
Lower	Upper	(1-α)	Lower	Upper
4603.69	4722.25	50%	4551.28	4774.67
4549.80	4776.15	75%	4472.39	4853.55
4490.52	4835.43	90%	4390.36	4935.59
4447.41	4878.54	95%	4338.15	4987.79
4355.79	4970.15	99%	4236.07	5089.88

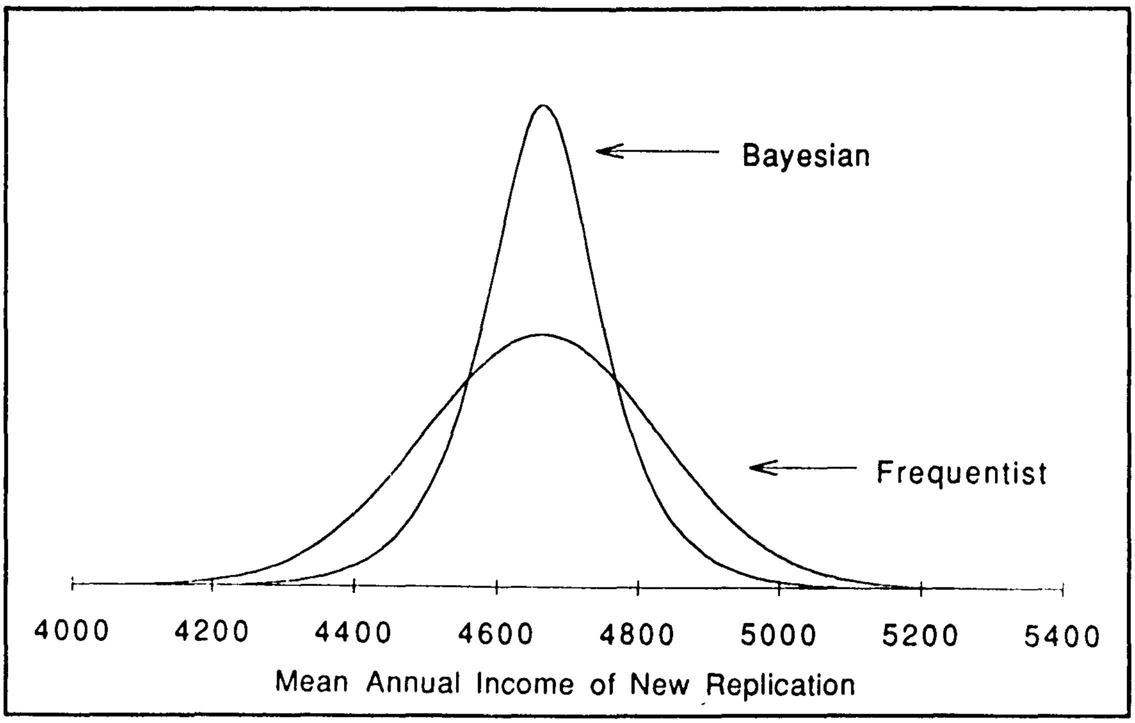

Figure 6.3 - Comparable Distributions

6.3.3 Using informative priors

Since the same simulation model is used here as in Section 6.2.3, the same values for the prior distribution parameters are used, as displayed in Table 6.7.

Table 6.14 shows the comparable means and standard deviations; and Table 6.15 shows the comparable intervals. Figure 6.4 shows the graphs of the comparable distributions.

The frequentist results are the same here as for the non-informative analysis. The Bayesian mean is a weighted average of the sample mean and prior mean. The Bayesian standard deviation is smaller than the frequentist standard deviation, resulting in narrower intervals. But, the Bayesian standard deviation here is larger than the Bayesian standard deviation from the non-informative analysis, as displayed in Table 6.12.

Table 6.14 - Comparable Descriptive Statistics

	Bayesian	Frequentist
mean	4542.34	4662.97
std.dev.	107.21	165.72

Table 6.15 - Comparable Intervals

Bayesian			Frequentist	
Lower	Upper	(1-α)	Lower	Upper
4475.71	4608.98	50%	4551.28	4774.67
4424.45	4660.24	75%	4472.39	4853.55
4368.06	4716.63	90%	4390.36	4935.59
4327.06	4757.63	95%	4338.15	4987.79
4239.92	4844.77	99%	4236.07	5089.88

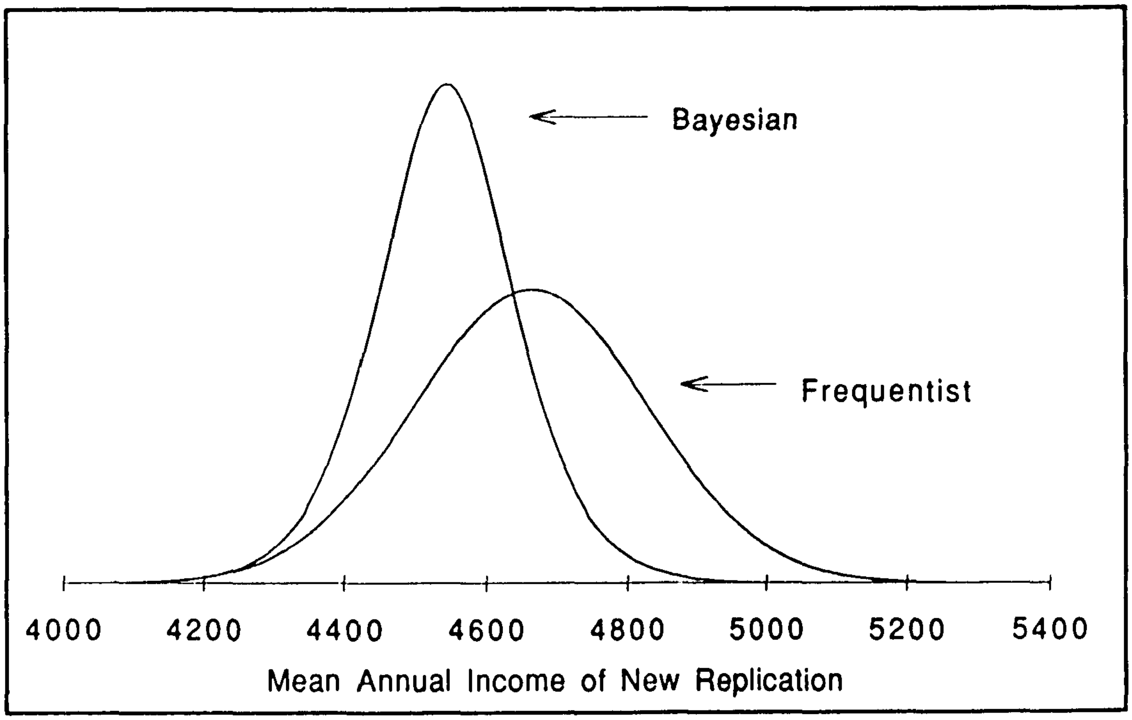

Figure 6.4 - Comparable Distributions

6.4 Nakamura Model Data Set, 400 Replications, Transposed; Type 3 Example

6.4.1 Data set description

In this section, the methodology is applied to the output data set from the first 400 replications of the Nakamura model. In order to obtain a Type 3 situation, the rows and columns are transposed; consequently, the variable of interest is the mean annual income over all replications for a randomly selected wife. (The reader is cautioned that this data set is being used in this manner only to demonstrate the approximation methodology when the row variance mode is at the zero boundary. This analysis does not adequately account for the proportion of wives in each replication who do not work; consequently, the large standard deviation results in a posterior distribution which gives a probability of negative earnings which is unrealistically high.)

Table 6.16 presents the sample descriptive statistics for this data set; the method-of-moments estimates for the variances are displayed in Table 6.17, note the negative value for the row variance estimate.

Table 6.16 - Sample Descriptive Statistics

Statistic	Value	Statistic	Value
I	400	MSR	226359015.06
J	1124	MSC	12347959155.68
$\bar{y}_{..}$	4662.97	MSE	231644954.28

Table 6.17
Method-of-Moments
Estimates of Variances

Variance	Value
Row	- 4702.79
Column	30290785.50
Error	231644954.28

6.4.2 Using non-informative Priors

The prior parameters for this analysis are presented in Table 6.4. Table 6.18 displays comparable means and standard deviations; and Table 6.19 displays comparable intervals. Figure 6.5 displays the graphs of the comparable distributions.

The means have the same value. The Bayesian standard deviation is larger than the frequentist standard deviation, resulting in wider intervals. But, since the difference in standard deviations is relatively small, the graphs of the distributions are not distinguishable given the resolution of the graphics device used here.

Table 6.18 - Comparable Descriptive Statistics

	Bayesian	Frequentist
mean	4662.97	4662.97
std.dev.	5508.69	5506.20

Table 6.19 - Comparable Intervals

Bayesian			Frequentist	
Lower	Upper	$(1-\alpha)$	Lower	Upper
1082.28	8243.66	50%	951.79	8374.15
- 1672.09	10998.04	75%	- 1669.16	10995.10
- 4426.47	13752.41	90%	- 4394.73	13720.68
- 6079.09	15405.04	95%	- 6129.18	15455.13
- 9659.79	18985.73	99%	- 9521.00	18846.95

6.4.3 Using informative priors

The prior parameters for this analysis are displayed in Table 6.20. Except for interchanging the row and column parameters to reflect the transposition of the rows and columns in the data set, the values of the prior parameters are the same as for the analysis in Section 6.2.3.

Table 6.20 - Prior Distribution Parameter Values

Parameter	Value	Parameter	Value
μ	4000.00	α_C	12.00
τ	100000.00	β_C	0.0001
α_R	4.00	α_E	2.50
β_R	0.0008	β_E	0.005

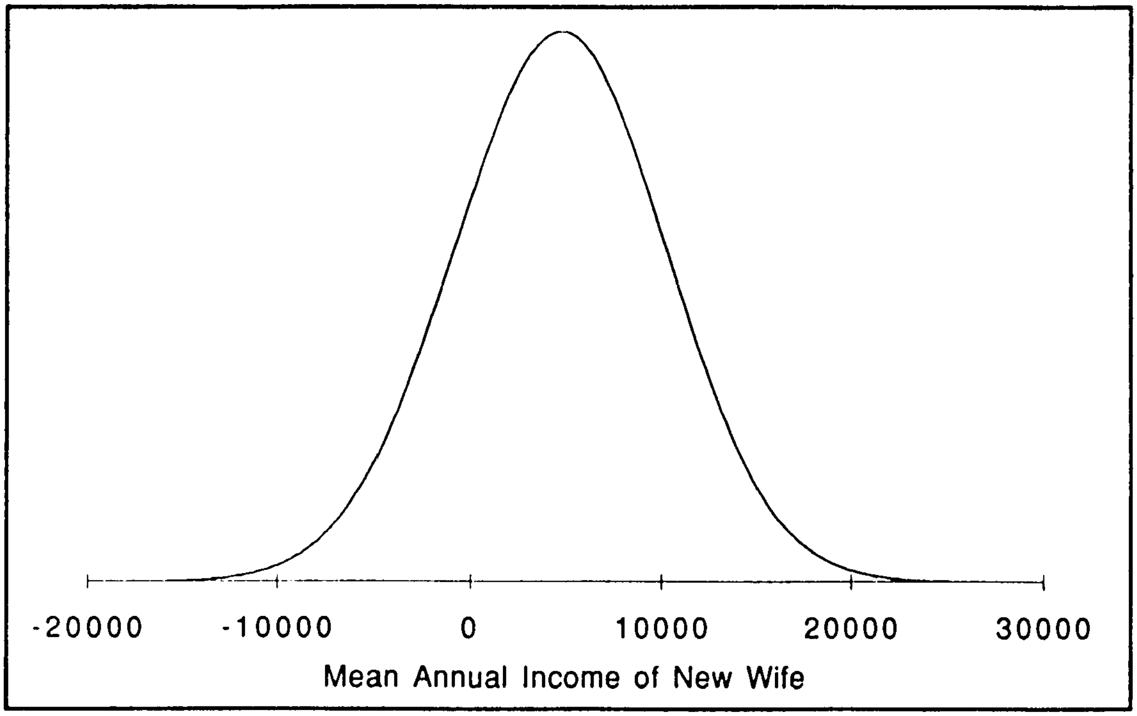

Figure 6.5 - Comparable Distributions

Table 6.21 displays the comparable means and standard deviations; and Table 6.22 displays the comparable intervals. Figure 6.6 displays the graphs of the comparable distributions.

The sampling theory results are the same as for the non-informative analysis of the previous section. The Bayesian mean is a weighted average of the sample mean and prior mean; the difference in the means is small compared to the magnitude of the standard deviations, so the difference in locations for the graphs of the distributions is barely distinguishable in Figure 6.6. The Bayesian standard deviation is smaller than the frequentist standard deviation, resulting in narrower intervals. Since the difference in the standard deviations is relatively small, the graphs of the distributions are not distinguishable given the resolution of the graphics device used.

Table 6.21 - Comparable Descriptive Statistics

	Bayesian	Frequentist
mean	4542.34	4662.97
std.dev.	5457.33	5506.20

Table 6.22 - Comparable Intervals

Bayesian			Frequentist	
Lower	Upper	(1-α)	Lower	Upper
995.02	8089.66	50%	951.79	8374.15
- 1733.68	10818.37	75%	- 1669.16	10995.10
- 4462.39	13547.08	90%	- 4394.73	13720.68
- 6099.61	15184.30	95%	- 6129.18	15455.13
- 9646.93	18731.62	99%	- 9521.00	18846.95

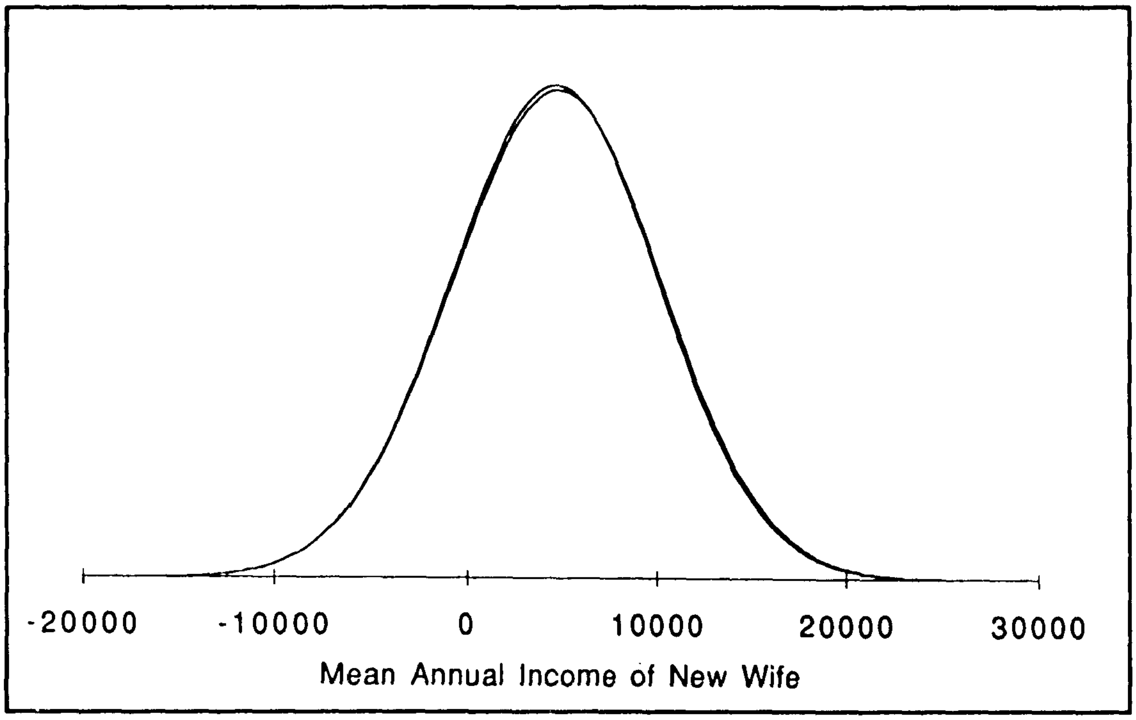

Figure 6.6 - Comparable Distributions

6.5 Type 4 Example

In this section, the methodology is applied to a sample data set which was chosen to achieve a Type 4 situation. The sample descriptive statistics are displayed in Table 6.23; the method-of-moments estimates for the variances are displayed in Table 6.24. Note the negative values for the row and column variance estimates.

Table 6.23 - Sample Descriptive Statistics

Statistic	Value	Statistic	Value
I	1000	MSR	0.00501
J	12000	MSC	0.00075
$\bar{y}_{..}$	0.00	MSE	1029922.23910

Table 6.24
Method-of-Moments
Estimates of Variances

Variance	Value
Row	- 85.8269
Column	- 1029.9222
Error	1029922.2391

The non-informative prior parameters for this analysis are presented in Table 6.4. Figure 6.7 displays the graph of the Bayesian posterior distribution. The approximation analysis does not work properly in this situation, as evidenced by the U shape for the posterior distribution. This distribution shape is typical of other examples for a Type 4 data set generated for this work, whether using informative or non-informative prior distributions.

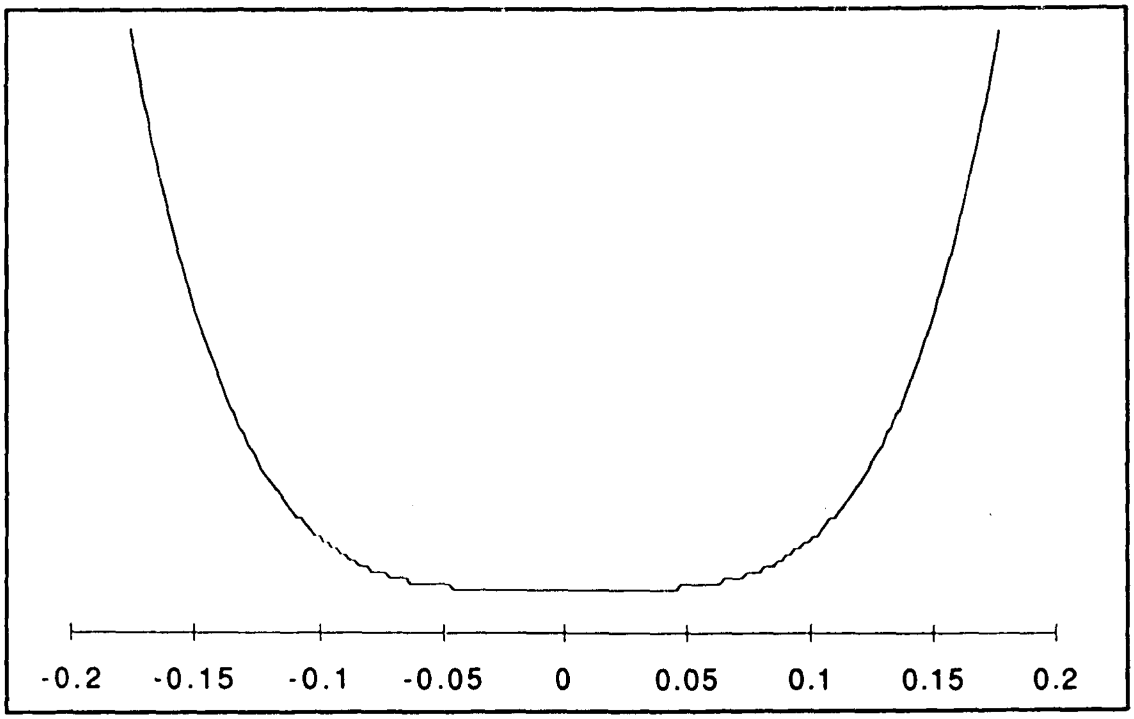

Figure 6.7 - Bayesian Posterior Distribution

6.6 Comments on the Approximation Methodology

In this section, some general comments are made reflecting the experiences of working with the approximation program. While the approximation methodology works for the data sets used in Sections 6.2 through 6.4, it does not work for all situations. First, problems encountered with various data sets are discussed. Then, some effects of different sample sizes are discussed.

The conditions for which the LaPlace approximation for integrals is applicable are described in detail in Kass, Tierney and Kadane (1990). One of the conditions is that the determinant of the Hessian matrix have positive value, since its square root is used in the denominator. Experience with the Nakamura model output, and arbitrary data sets generated from the two-way random effects model, shows that is not always the case for the three functions used in this analysis, as given in Section 5.1.2 and Expression (5.6) in general. For example, when approximating the integral in the denominator of the variance using the sample data from the first 10 replications of the Nakamura model and diffuse priors the determinant of the Hessian matrix has a negative value.

Another problem encountered which causes the approximation methodology to break down is the occurrence of negative values for the arguments of the logarithms in the resulting integrand after having been approximated using Result (5.7) for situations with the row and/or column variance mode at the zero boundary. This can occur in Types 2, 3, or 4 situations, in the numerators of Equations (5.12), (5.14), or (5.16), respectively. For example, when evaluating the posterior distribution at the posterior mean using the sample data from the first 500 replications of the Nakamura model and diffuse priors results in a negative value for the argument of the logarithm function.

The problems described above are not attributable solely to sample size, since the approximation methodology can work for small sample sizes. The LaPlace method for integral approximation is based upon asymptotic arguments, as the value $t \to \infty$, see Sections 5.2.2, 5.2.3 and 5.2.4. In the two-way random effects model application, this condition corresponds to having $I \to \infty$ and $J \to \infty$ at the same time. It is not merely the sample size which determines whether or not the approximation works, but the entire data set configuration, meaning the sample descriptive statistics and prior parameter values taken together. In terms of the general function for the integrands described in Section 5.1.3, the values of the exponents W1 through W10 are determinative of the success of the approximation.

For an example of an application using a small sample, the methodology is applied to the data set taken from Table 6.2.3, "Average mileage for 9 drivers on 9 cars", Box and Tiao (1973, p. 336). The mean fuel economy (mpg) over all cars for a randomly selected driver is the variable of interest. The descriptive statistics for this sample data set are presented in Table 6.25; the method-of-moments estimates for the variances are displayed in Table 6.26.

Table 6.25 - Sample Descriptive Statistics

Statistic	Value	Statistic	Value
I	9	MSR	63.23
J	9	MSC	22.63
$\bar{y}_{..}$	27.72	MSE	0.86

Table 6.26
Method-of-Moments
Estimates of Variances

Variance	Value
Row	6.93
Column	2.42
Error	0.86

The non-informative prior parameters for this analysis are presented in Table 6.4. Table 6.27 shows comparable means and standard deviations; and Figure 6.8 shows the graphs of the comparable distributions.

Table 6.27 - Comparable Descriptive Statistics

	Bayesian	Frequentist
mean	27.72	27.72
std.dev.	2.06	1.86

Sample size does influence the behavior of the approximations since as sample size increases the modes of the variances tend to stabilize, in the sense that they stay at the same set of values for more of the evaluations. In each analysis, the approximation is performed 103 times: once each for the variance numerator, variance denominator, and posterior distribution evaluated at the posterior mean; and for the posterior distribution evaluated at 100 points evenly spaced over 5 standard deviations above the

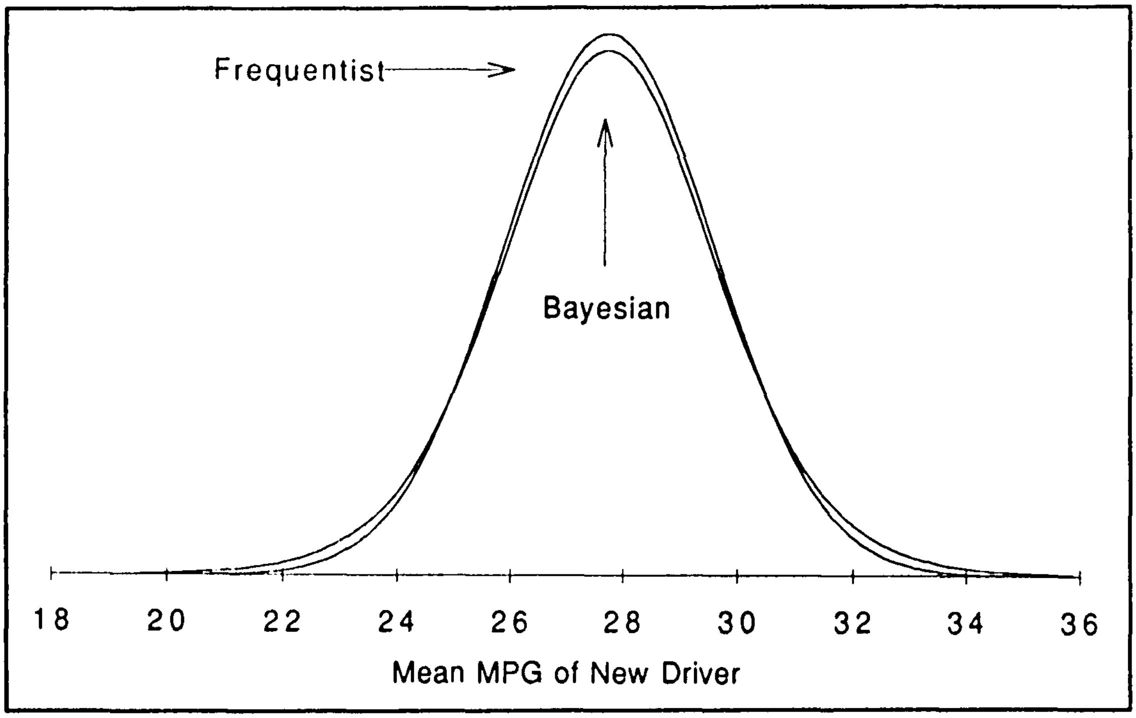

Figure 6.8 - Comparable Distributions

mean. Due to the symmetry of the distribution, the value of the function at a point below the mean is set equal to the value of the function at the corresponding point above the mean, rather than again performing the approximation. For the Box and Tiao data set just described, a different set of modes occurs for each of the 103 approximations. Using the first 10 replications of the Nakamura model data set with non-informative priors, 5 different sets of modes occur: one set of modes occurs for the variance numerator and denominator approximations, another set of modes occurs for the posterior distribution at the posterior mean through the 46th point above the mean, and different sets of modes for the 47th through 66th points, 67th thorough 83rd points, and 84th through 100th points. And, using the 1000 replications of the Nakamura model data set as described in Section 6.2.2 the same set of modes occurs for all 103 approximations.

The effect of different sets of modes occurring for the successive approximations may or may not be apparent in the resulting posterior distributions. For example, the graph of the Bayesian posterior distribution for the Box and Tiao data set in Figure 6.8 appears quite smooth even though each approximation uses a different set of modes. Contrast the posterior distribution using the first 10 replications of the Nakamura model with non-informative priors, shown in Figure 6.9. Careful inspection of the upper tail reveals two jumps in the distribution. These jumps occur at the 67th and 84th points, where the sets of modes change, but no discernible jump occurs at the 47th point where another change in the set of modes occurs. An enlarged display of part of the upper tail is presented in Figure 6.10; the diamonds represent the discrete set of approximations to the continuous posterior distribution. The jumps in the posterior distribution corresponding to the changes in the sets of modes at the 67th and 84th points are more apparent in this figure than in the earlier figure.

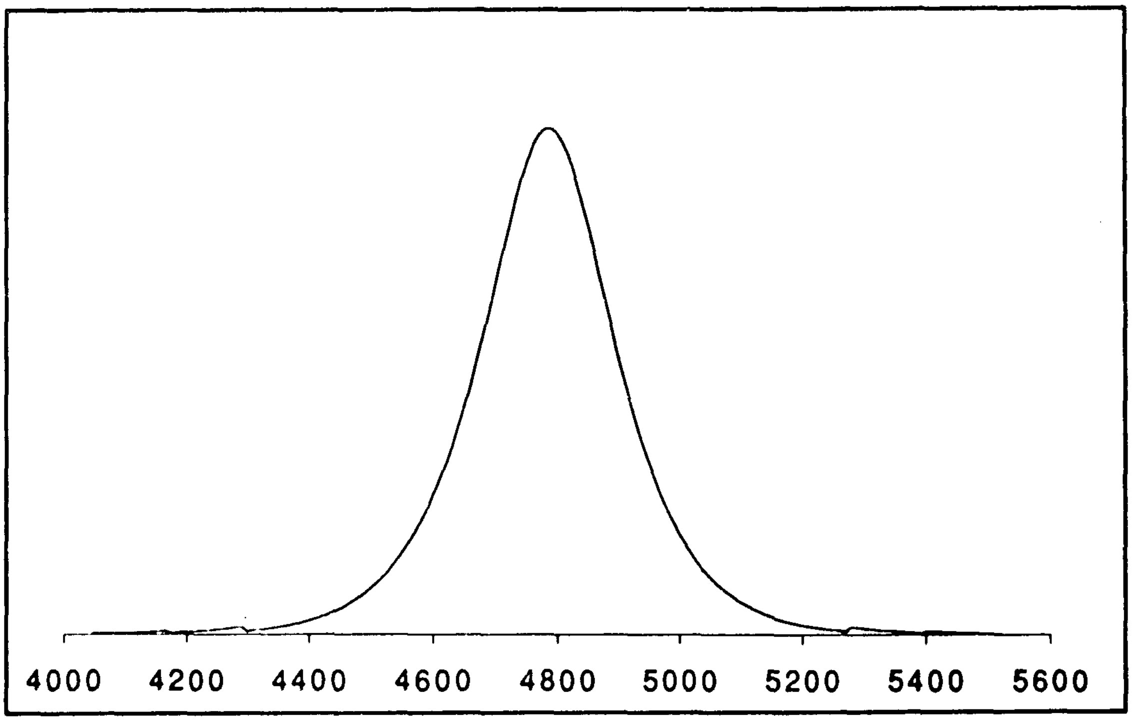

Figure 6.9 - Bayesian Posterior Distribution

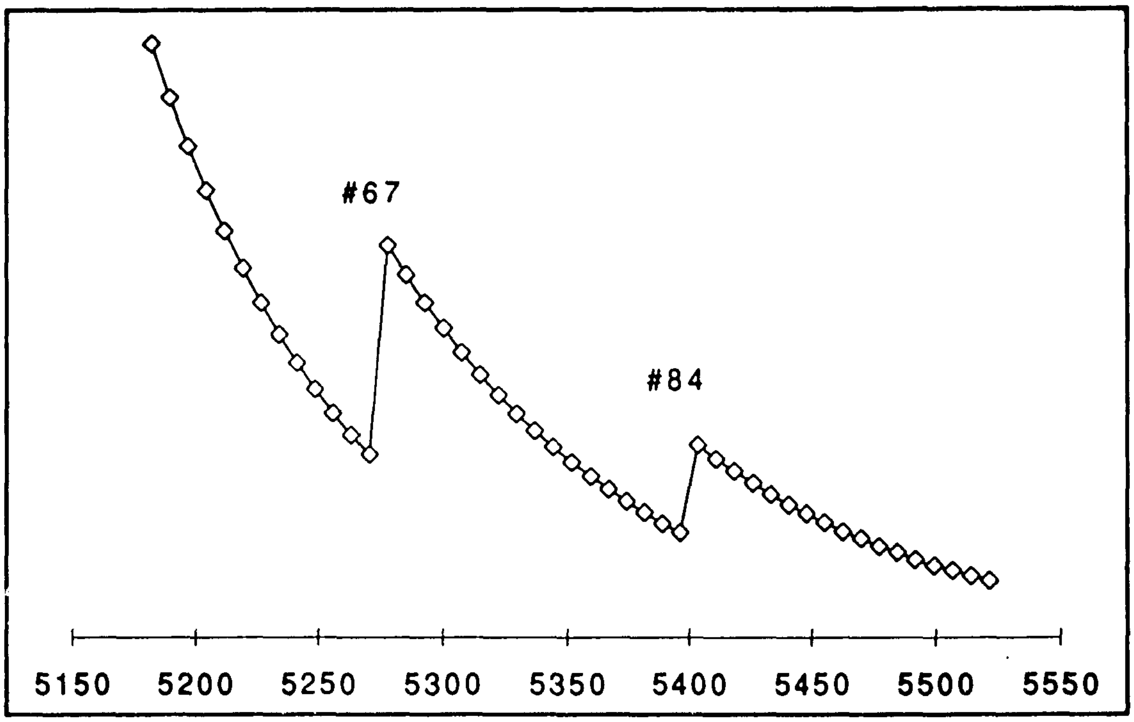

Figure 6.10 - Upper Tail: Points 54 to 100

CHAPTER 7
CONCLUSIONS AND EXTENSIONS

7.1 Conclusions

This work explores the use of a two-way random effects model as an appropriate metamodel for microsimulation output analysis. Comparisons between sampling theory and Bayesian predictive distributions for the mean of an unobserved replication of the simulation model are made, using as examples output from a simulation model of the labor force participation and incomes of wives based on Nakamura and Nakamura (1985a).

This work explains how the use of a metamodel enhances the analysis of output from a microsimulation model. The two-way random effects model is an appropriate metamodel for many uses of microsimulation models. The output of the microsimulation model, consisting of measurements on the dependent variable for each decision unit over a number of independent replications, is matched by the structure of the two-way random effects model. The use of this metamodel permits the inherent variability of microsimulation models to be identified, separated and investigated. Using this metamodel to make predictive inference about the mean response of an unobserved replication of the microsimulation model focuses the attention of the model user on an appropriate measure matching the behavior of the real system being studied.

This work demonstrates the advantages of Bayesian analysis over sampling theory analysis since the former permits the incorporation of the model user's experience and knowledge into the analysis and provides a coherent method of dealing with sample data which results in negative estimated values for model variance parameters which by definition may only assume positive values.

The difficulty encountered in Bayesian analysis of intractable integrations over nuisance parameters is addressed by using LaPlace's method of integral approximation. While the analytic-numeric approximations developed in this work using LaPlace's method may not be applicable to all problem sets, they are shown to be useful for approximating the predictive distribution of the mean of an unobserved replication of the Nakamura microsimulation model for the large number of replications used here.

7.2 Extensions

There are a number of directions in which future research can go; the variety of topics include the areas of economics, simulation, statistics, and mathematics.

With simulation models of income for economic units, model users may be interested in a number of other descriptive measures of the distribution of incomes. This work explored the posterior distribution of the mean of an unobserved replication of the model, as a proxy for the mean of the simulated world in an unobserved setting, that is, in the future or under some other set of operating characteristics. Other descriptive measures of income distributions may include percentiles, proportions, or more complex measures such as the Lorenz curve or its Gini coefficient.

Other metamodels may be needed for the analyses when interest is on other measures of system performance, or when other probability distributions are more appropriate than the normal distribution for describing system behavior.

Percentiles of the distribution of incomes describe the income level of certain groups of the population when the members are ranked by income. The model user specifies the percentage groupings of the population, and the dependent variable is the dollar value for which that percent of the population has income at or below. Some examples which may be of particular interest are the median, a univariate measure, or the first and fourth quintiles, the 20^{th} and 80^{th} percentiles, multivariate measures. The posterior distribution of percentiles is based on the order statistics; a Bayesian analysis for simulation output analysis may be based on the work of Hill (1968).

Proportions of the population which have income in various classes may be of interest to the model user. The user specifies the class boundaries of interest, and the dependent variable is proportion of the population in each category. If a single income level is specified, such as a poverty level, then the analysis is based upon the binomial distribution. If two or more income classes are specified, then the analysis is multivariate. The articles by Leonard (1972, 1975) and Lenk (1990) may provide a starting point for these analyses. Andrews, Birdsall, Gentner, and Spivey (1986) addressed the trinomial distribution with categories of incomes below \$25,000, from \$25,000 to \$50,000, and above \$50,000.

The Lorenz curve is a description of the income distribution based on quantile-quantile plots. When the population elements are ranked on wealth, the curve depicts the proportion of cumulative wealth owned by the cumulative proportion of the population. The Gini coefficient is a numerical description of the discrepancy between the Lorenz curve for a given wealth distribution and the curve resulting from an equal distribution of wealth.

Other multivariate measures of income which may be of interest to model users include the joint distribution of the mean and standard deviation of income, or the joint distribution of the mean of incomes for those employed and the proportion of unemployed.

An issue of major concern to the simulation modeler as well as the model user is model validation; Andrews, et al. (1986) addressed the validation of microsimulation models. For microsimulation models, an appropriate validation technique is *ex post forecasting*. This method uses a sequence of historic data, split at some time point in the past. The data prior to the break point is used to estimate the coefficient values for the model. Then, the simulation model is replicated forward in time from the break point and the model output compared to the historic data for the corresponding time periods. The validation question then is to describe the difference in the inference one makes using the simulation replications from the inference one makes using the historic data.

The decision unit sample design determines the analysis methodology. The two-way random effects metamodel and the corresponding sampling theory and Bayesian predictive intervals developed in this work assumed the decision unit was a simple random sample; this is why the SRC subsample from the PSID was used rather than the entire PSID sample. Andrews, et al. (1986) addressed the complex sample design issue, in particular dealing with the stratified, paired-cluster design of the full PSID.

The metamodel in this work assumes that the error terms are independent over all decision units and replications. In other simulation models, such as those of queuing systems, there is dependency among the model units within a replication. Such dependency may be modeled by using error terms which are correlated within each replication. The two-way random effects model with error terms correlated within a replication and independent across replications may be an appropriate metamodel for a terminating condition queuing system, with multiple observations of the system obtained by the method of replications. Tiao and Tan (1966) addresses the effect of autocorrelated errors in the random effects models.

Issues concerning the approximation of the intractable integrals arising in the Bayesian predictive distribution analysis for the mean of the $(J+1)^{th}$ replication need further investigation. In describing the regularity conditions sufficient for validity of the LaPlace approximation, Kass, Tierney and Kadane (1990) do not establish a general theorem, but intend that the regularity conditions "be verified for interesting special families as the need arises in practice." The issues, then, are what the regularity conditions imply about the data configuration, that is, the sample data and prior parameter values, such that the LaPlace approximations will be valid for the two-way random effects metamodel for microsimulation models. An alternative methodology which may prove useful for the problems discussed in this work is the use of Gibbs sampling; this methodology would be used, in lieu of the LaPlace methods described in Sections 5.3 through 5.6, to obtain the predictive distribution, from Equation (4.13).

From the array of available topics, the development of Gibbs sampling procedures will be the first topic investigated, followed by an investigation of the validation of microsimulation models.

APPENDICES

APPENDIX A

COMPUTER COMMAND SEQUENCES

This appendix lists computer command sequences used in the processing of data from the PSID tapes.

A.1 Command Sequence for Extracting Decision Unit Sample From PSID Tapes

```
 1   $mount
 2   157-lisp-167 *t1*
 3   157-cusp-167 *t2*
 4   $endfile
 5   $run isr:osiris.iv
 6   &trans dictin=*t1*(fi=1) datain=*t1*+*t2*(fi=2) -
          dictout=-dict dataout=-data
 7   include v5336=0001-3000 and v5852=29-63 and v5650=1 -
          and v6219=1 and v6812=1
 8   title
 9   v=5203,5353,5703,5743,5788,58525854,6116,6123,6174, -
          6209,6302,6348,6398,6767
10   &end
11   &stop
12   $create psiddata type=seq
13   $run stat:midas
14   osiris var=all max=1802 case=all fi=-dict;-data &
          option=none
15   write internal v=all fi=psiddata
16   code v1=cuts(v6116) points=,99, lab=*
17   code v2=cuts(v6123) points=,99, lab=*
18   trans v6303=v6302(-1) lab=lag79id
19   trans v6303=replace(1.) lab=*
20   trans v6304=v6302:v6303 lab=*
21   code v6305=ordinal(v6304) lab=*
22   write internal special fi=psiddata  &
          cases=v1:1*v2:1*v6305:1 &
          var=5203,5353,5703,5743,5788,5852-5854,6116,6123, &
          6174,6209,6348,6398,6767
23   finish
```

A.2 Command Sequence for Moving Decision Unit Sample from MIDAS INTERNAL File to FORTRAN Formatted Data File

```
1    $create seeddata
2    $run stat:midas
3    read internal v=all fi=psiddata
4    write file=seeddata format=(i1,8i2,i4,i5,2i6) case=all &
          var=6209,5203,5703,5353,5853,5852,5854,6116,6123, &
          5743,5788,6174,6767
5    finish
```

APPENDIX B

SIMULATION MODEL PROGRAM

Table B.1 - Input / Output Device Designation

#	Use	Description
1	Input	Decision Unit Sample
2	Input / Output	PRNG Seeds
3	Output	Earnings Vector
5	Input	*SOURCE*
6	Output	*SINK*

```
C*****************************************************
C  FORTRAN program to perform simulation of working wives
C  using DIFFERENCE model in Nakamura, A. and Nakamura, M.
C  (1985), "Dynamic Models of the Labor Force Behavior of
C  Married Women Which Can Be Estimated Using Limited
C  Amounts of Past Information," JOURNAL OF ECONOMETRICS,
C  27, 273-298.
C*****************************************************
C  declaration section
C
      INTEGER M2INDX(9),M3INDX(7)
      REAL*8 M1BETA(17,4),M1CNST(4),M2BETA(10,4),M2CNST(4),
     &       M2STDV(4),M3BETA(10,4),M3CNST(4),M3STDV(4)
      COMMON/MODEL/ M2INDX,M3INDX,M1BETA,M1CNST,M2BETA,
     &              M2CNST,M2STDV,M3BETA,M3CNST,M3STDV
      REAL*8 CPI77,CPI78,RATE77(51),RATE78(51),WAGE77(51),
     &       WAGE78(51)
      COMMON/MACRO/ CPI77,CPI78,RATE77,RATE78,WAGE77,WAGE78
C
      INTEGER ISEED,NWIVES,WIFE,COL,AGE,I
      LOGICAL WORKED,YOUNG
      REAL*8 PI,WINDEX(2000),M2DIST(2000),M3DIST(2000),
     &X(17),PROBIT,CDF,PDF,LAMBDA,OFWGRT,HRSWRK,WFEARN(2000)
C
C  initialize values
C
      PI = DCONST('PI')
C
      CALL MACROV
      CALL MODVAL
C
      READ(1,199)NWIVES
  199 FORMAT(I4)
C
C  SEED value obtained from, and returned to, file on #3;
C  vector of values of U(0,1) R.V.s for probability of
C  working; 2 vectors of values of Standard Normal R.V. for
C  wage rate and hours predictions;
C
```

```
      READ(2,888)ISEED
  888 FORMAT(I10)
      CALL RNSET(ISEED)
      CALL DRNUN(NWIVES,WINDEX)
      CALL DRNNOR(NWIVES,M2DIST)
      CALL DRNNOR(NWIVES,M3DIST)
      CALL RNGET(ISEED)
      WRITE(2,888)ISEED
C
      DO 1000 WIFE=1,NWIVES
C
C  input section
C
       CALL INDATA(WORKED,YOUNG,X)
C
C  determine which column from tables to use
C
       IF (WORKED) THEN
        IF (YOUNG) THEN
         COL = 1
        ELSE
         COL = 2
        ENDIF
       ELSE
        IF (YOUNG) THEN
         COL = 3
        ELSE
         COL = 4
        ENDIF
       ENDIF
C
C  calculate probit index
C
       PROBIT = M1CNST(COL)
       DO 101 I=1,17
  101  PROBIT = PROBIT+M1BETA(I,COL)*X(I)
C
C  calculate probability of working this year
C
       CDF = DNORDF(PROBIT)
C
C  stochastic determination if working this year
C
       IF (CDF.LT.WINDEX(WIFE)) THEN
        WFEARN(WIFE) = 0.D0
        GO TO 1000
       ENDIF
C
C  calculate selection bias term
C
       PDF = DEXP(-(PROBIT*PROBIT)/2.D0)/DSQRT(2.D0*PI)
       LAMBDA = PDF/CDF
C
C  calculate predicted log of offered wage rate
C
       OFWGRT = M2CNST(COL)
       DO 201 I=1,9
  201  OFWGRT = OFWGRT+M2BETA(I,COL)*X(M2INDX(I))
```

```
         OFWGRT = OFWGRT+M2BETA(10,COL)*LAMBDA
     &                  +M2DIST(WIFE)*M2STDV(COL)
C
C  calculate predicted log of annual hours of work
C
         HRSWRK = M3CNST(COL)+M3BETA(1,COL)*OFWGRT
     &                       +M3BETA(2,COL)*OFWGRT
         DO 301 I=3,9
  301    HRSWRK = HRSWRK+M3BETA(I,COL)*X(M3INDX(I-2))
         HRSWRK = HRSWRK+M3BETA(10,COL)*LAMBDA
     &                  +M3DIST(WIFE)*M3STDV(COL)
C
C  calculate earnings
C
         IF (WORKED) THEN
          OFWGRT = X(2)+OFWGRT
          HRSWRK = X(1)+HRSWRK
         ENDIF
         WFEARN(WIFE) = DEXP(OFWGRT+HRSWRK)
         IF (WFEARN(WIFE).GT.999999.D0) WFEARN(WIFE)=999999.D0
C
C  end loop
C
 1000 CONTINUE
C
C  output earnings
C
  900 WRITE(3,990) (WFEARN(I), I=1,NWIVES)
  990 FORMAT(200F7.0)
      STOP
      END
C*********************************************************
C  subroutine to input values of model var's for each wife
C*********************************************************
      SUBROUTINE INDATA(WORKED,YOUNG,X)
C
C  declaration section
C
      INTEGER AGE,EDUCTN,HEDINC(2),NRKIDS(2),RACE,STATE(2),
     &        WIFHRS,WIFINC,YRSWRK,YNGEST
      LOGICAL WORKED,YOUNG
      REAL*8 X(17)
      REAL*8 CPI77,CPI78,RATE77(51),RATE78(51),WAGE77(51),
     &       WAGE78(51)
      COMMON/MACRO/ CPI77,CPI78,RATE77,RATE78,WAGE77,WAGE78
C
      READ(1,999) RACE,STATE,NRKIDS,AGE,YNGEST,EDUCTN,
     &            YRSWRK,WIFHRS,WIFINC,HEDINC
  999 FORMAT(I1,8I2,I4,I5,2I6)
C
      IF(AGE.LT.47) THEN
       YOUNG = .TRUE.
      ELSE
       YOUNG = .FALSE.
      ENDIF
      IF((WIFHRS.GT.0).AND.(WIFINC.GT.0)) THEN
       WORKED = .TRUE.
      ELSE
```

```
       WORKED = .FALSE.
      ENDIF
C
      IF (WORKED) THEN
       X(1) = DLOG(DFLOAT(WIFHRS))
       X(2) = DLOG(DFLOAT(WIFINC)/(CPI77*DFLOAT(WIFHRS)))
      ELSE
       X(1) = 0.D0
       X(2) = 0.D0
      ENDIF
      X(3) = DFLOAT(YRSWRK)/(DFLOAT(AGE)-17.D0)
      IF (YRSWRK.EQ.0) THEN
       X(4) = 1.D0
      ELSE
       X(4) = 0.D0
      ENDIF
      IF ((NRKIDS(2).GT.NRKIDS(1)).AND.(YNGEST.EQ.1)) THEN
       X(5) = 1.D0
      ELSE
       X(5) = 0.D0
      ENDIF
      IF ((YNGEST.GT.1).AND.(YNGEST.LT.6)) THEN
       X(6) = 1.D0
      ELSE
       X(6) = 0.D0
      ENDIF
      X(7) = DFLOAT(NRKIDS(2))
      X(8) = DFLOAT(AGE)
      X(9) = DFLOAT(EDUCTN)
      IF (RACE.EQ.2) THEN
       X(10) = 1.D0
      ELSE
       X(10) = 0.D0
      ENDIF
      X(11) = DFLOAT(HEDINC(2))/(1000.D0*CPI78)
      X(12) = X(11)-DFLOAT(HEDINC(1))/(1000.D0*CPI77)
      IF (X(12).LT.0.D0) THEN
       X(13) = X(12)
      ELSE
       X(13) = 0.D0
      ENDIF
      X(14) = WAGE78(STATE(2))/CPI78
      X(15) = X(14)-WAGE77(STATE(1))/CPI77
      X(16) = RATE78(STATE(2))
      X(17) = X(16)-RATE77(STATE(1))
C
      RETURN
      END
C***********************************************************
C  subroutine to assign values of macro-economic variables
C  from HANDBOOK OF LABOR STATISTICS, Bulletin #2070
C***********************************************************
      SUBROUTINE MACROV
C
C  declaration section
C
      REAL*8 CPI77,CPI78,RATE77(51),RATE78(51),WAGE77(51),
     &       WAGE78(51)
```

```
      COMMON/MACRO/ CPI77,CPI78,RATE77,RATE78,WAGE77,WAGE78
C
C  Consumer Price Index (1967$ = 100); source:  Table 134
C
      CPI77 = 1.815D0
      CPI78 = 1.954D0
C
C  state unemployment rate, 1977; source:  Table 45
C
      RATE77(1)  = 7.4D0
      RATE77(2)  = 8.2D0
      RATE77(3)  = 6.6D0
      RATE77(4)  = 8.2D0
      RATE77(5)  = 7.0D0
      RATE77(6)  = 6.2D0
      RATE77(7)  = 8.4D0
      RATE77(8)  = 9.7D0
      RATE77(9)  = 8.2D0
      RATE77(10)  = 6.9D0
      RATE77(11)  = 5.9D0
      RATE77(12)  = 6.2D0
      RATE77(13)  = 5.7D0
      RATE77(14)  = 4.0D0
      RATE77(15)  = 4.1D0
      RATE77(16)  = 4.7D0
      RATE77(17)  = 7.0D0
      RATE77(18)  = 8.4D0
      RATE77(19)  = 6.1D0
      RATE77(20)  = 8.1D0
      RATE77(21)  = 8.2D0
      RATE77(22)  = 5.1D0
      RATE77(23)  = 7.4D0
      RATE77(24)  = 5.9D0
      RATE77(25)  = 6.4D0
      RATE77(26)  = 3.7D0
      RATE77(27)  = 7.0D0
      RATE77(28)  = 5.9D0
      RATE77(29)  = 9.4D0
      RATE77(30)  = 7.8D0
      RATE77(31)  = 9.1D0
      RATE77(32)  = 5.9D0
      RATE77(33)  = 4.8D0
      RATE77(34)  = 6.5D0
      RATE77(35)  = 5.0D0
      RATE77(36)  = 7.4D0
      RATE77(37)  = 7.7D0
      RATE77(38)  = 8.6D0
      RATE77(39)  = 7.2D0
      RATE77(40)  = 3.3D0
      RATE77(41)  = 6.3D0
      RATE77(42)  = 5.3D0
      RATE77(43)  = 5.3D0
      RATE77(44)  = 7.0D0
      RATE77(45)  = 5.3D0
      RATE77(46)  = 8.8D0
      RATE77(47)  = 7.1D0
      RATE77(48)  = 4.9D0
      RATE77(49)  = 3.6D0
```

```
      RATE77(50) = 9.4D0
      RATE77(51) = 7.3D0
C
C  state unemployment rate, 1978; source:  Table 45
C
      RATE78(1) = 6.3D0
      RATE78(2) = 6.1D0
      RATE78(3) = 6.3D0
      RATE78(4) = 7.1D0
      RATE78(5) = 5.2D0
      RATE78(6) = 5.5D0
      RATE78(7) = 7.6D0
      RATE78(8) = 8.5D0
      RATE78(9) = 6.6D0
      RATE78(10) = 5.7D0
      RATE78(11) = 5.7D0
      RATE78(12) = 6.1D0
      RATE78(13) = 5.7D0
      RATE78(14) = 4.0D0
      RATE78(15) = 3.1D0
      RATE78(16) = 5.2D0
      RATE78(17) = 7.0D0
      RATE78(18) = 6.1D0
      RATE78(19) = 5.6D0
      RATE78(20) = 6.1D0
      RATE78(21) = 6.9D0
      RATE78(22) = 3.8D0
      RATE78(23) = 7.1D0
      RATE78(24) = 5.0D0
      RATE78(25) = 6.0D0
      RATE78(26) = 2.9D0
      RATE78(27) = 4.4D0
      RATE78(28) = 3.8D0
      RATE78(29) = 7.2D0
      RATE78(30) = 5.8D0
      RATE78(31) = 7.7D0
      RATE78(32) = 4.3D0
      RATE78(33) = 4.6D0
      RATE78(34) = 5.4D0
      RATE78(35) = 3.9D0
      RATE78(36) = 6.0D0
      RATE78(37) = 6.9D0
      RATE78(38) = 6.6D0
      RATE78(39) = 5.7D0
      RATE78(40) = 3.1D0
      RATE78(41) = 5.8D0
      RATE78(42) = 4.8D0
      RATE78(43) = 3.8D0
      RATE78(44) = 5.7D0
      RATE78(45) = 5.4D0
      RATE78(46) = 6.8D0
      RATE78(47) = 6.3D0
      RATE78(48) = 5.1D0
      RATE78(49) = 3.3D0
      RATE78(50) = 11.2D0
      RATE78(51) = 7.7D0
C
C  state average hourly wage in manufacturing, 1977;
```

```
C  source:  Table 97
C
      WAGE77(1)  =  4.89D0
      WAGE77(2)  =  5.55D0
      WAGE77(3)  =  4.30D0
      WAGE77(4)  =  6.00D0
      WAGE77(5)  =  5.56D0
      WAGE77(6)  =  5.80D0
      WAGE77(7)  =  5.94D0
      WAGE77(8)  =  5.50D0
      WAGE77(9)  =  4.63D0
      WAGE77(10)  =  4.46D0
      WAGE77(11)  =  5.82D0
      WAGE77(12)  =  6.28D0
      WAGE77(13)  =  6.60D0
      WAGE77(14)  =  6.43D0
      WAGE77(15)  =  5.11D0
      WAGE77(16)  =  5.69D0
      WAGE77(17)  =  5.75D0
      WAGE77(18)  =  4.52D0
      WAGE77(19)  =  6.05D0
      WAGE77(20)  =  5.13D0
      WAGE77(21)  =  7.54D0
      WAGE77(22)  =  5.97D0
      WAGE77(23)  =  4.15D0
      WAGE77(24)  =  5.75D0
      WAGE77(25)  =  6.53D0
      WAGE77(26)  =  5.39D0
      WAGE77(27)  =  6.10D0
      WAGE77(28)  =  4.56D0
      WAGE77(29)  =  5.80D0
      WAGE77(30)  =  4.43D0
      WAGE77(31)  =  5.67D0
      WAGE77(32)  =  4.10D0
      WAGE77(33)  =  5.19D0
      WAGE77(34)  =  6.74D0
      WAGE77(35)  =  5.31D0
      WAGE77(36)  =  6.67D0
      WAGE77(37)  =  5.85D0
      WAGE77(38)  =  4.39D0
      WAGE77(39)  =  4.28D0
      WAGE77(40)  =  4.84D0
      WAGE77(41)  =  4.68D0
      WAGE77(42)  =  5.42D0
      WAGE77(43)  =  5.18D0
      WAGE77(44)  =  4.70D0
      WAGE77(45)  =  4.69D0
      WAGE77(46)  =  6.83D0
      WAGE77(47)  =  6.06D0
      WAGE77(48)  =  6.16D0
      WAGE77(49)  =  5.70D0
      WAGE77(50)  =  9.12D0
      WAGE77(51)  =  5.51D0
C
C  state average hourly wage in manufacturing, 1978;
C  source:  Table 97
C
      WAGE78(1)  =  5.40D0
```

```
      WAGE78(2)  = 6.03D0
      WAGE78(3)  = 4.72D0
      WAGE78(4)  = 6.43D0
      WAGE78(5)  = 5.96D0
      WAGE78(6)  = 6.21D0
      WAGE78(7)  = 6.58D0
      WAGE78(8)  = 6.72D0
      WAGE78(9)  = 5.07D0
      WAGE78(10)  = 4.88D0
      WAGE78(11)  = 6.53D0
      WAGE78(12)  = 6.76D0
      WAGE78(13)  = 7.17D0
      WAGE78(14)  = 7.00D0
      WAGE78(15)  = 5.64D0
      WAGE78(16)  = 6.26D0
      WAGE78(17)  = 6.42D0
      WAGE78(18)  = 4.91D0
      WAGE78(19)  = 6.46D0
      WAGE78(20)  = 5.54D0
      WAGE78(21)  = 8.13D0
      WAGE78(22)  = 6.44D0
      WAGE78(23)  = 4.56D0
      WAGE78(24)  = 6.21D0
      WAGE78(25)  = 7.81D0
      WAGE78(26)  = 5.83D0
      WAGE78(27)  = 6.54D0
      WAGE78(28)  = 4.93D0
      WAGE78(29)  = 6.20D0
      WAGE78(30)  = 4.79D0
      WAGE78(31)  = 6.08D0
      WAGE78(32)  = 4.47D0
      WAGE78(33)  = 5.55D0
      WAGE78(34)  = 7.29D0
      WAGE78(35)  = 5.81D0
      WAGE78(36)  = 7.23D0
      WAGE78(37)  = 6.37D0
      WAGE78(38)  = 4.71D0
      WAGE78(39)  = 4.66D0
      WAGE78(40)  = 5.19D0
      WAGE78(41)  = 5.13D0
      WAGE78(42)  = 5.88D0
      WAGE78(43)  = 5.68D0
      WAGE78(44)  = 5.10D0
      WAGE78(45)  = 5.11D0
      WAGE78(46)  = 7.56D0
      WAGE78(47)  = 6.68D0
      WAGE78(48)  = 6.69D0
      WAGE78(49)  = 6.18D0
      WAGE78(50)  = 8.86D0
      WAGE78(51)  = 5.90D0
C
      RETURN
      END
C*************************************************************
C  subroutine to assign values of model coefficients
C*************************************************************
      SUBROUTINE MODVAL
C
```

```
C  declaration section
C
      INTEGER M2INDX(9),M3INDX(7)
      REAL*8 M1BETA(17,4),M1CNST(4),M2BETA(10,4),M2CNST(4),
     &       M2STDV(4),M3BETA(10,4),M3CNST(4),M3STDV(4)
      COMMON/MODEL/ M2INDX,M3INDX,M1BETA,M1CNST,M2BETA,
     &              M2CNST,M2STDV,M3BETA,M3CNST,M3STDV
C
C  probit index model estimated coefficients from Table A.1
C
      M1CNST(1) = 0.345D0
      M1CNST(2) = -1.984D0
      M1CNST(3) = 0.530D0
      M1CNST(4) = 1.997D0
C
      M1BETA(1,1)  = 0.289D0
      M1BETA(2,1)  = 0.406D0
      M1BETA(3,1)  = -0.015D0
      M1BETA(4,1)  = 0.D0
      M1BETA(5,1)  = -0.272D0
      M1BETA(6,1)  = 0.335D0
      M1BETA(7,1)  = 0.027D0
      M1BETA(8,1)  = 0.017D0
      M1BETA(9,1)  = -0.008D0
      M1BETA(10,1)  = -0.217D0
      M1BETA(11,1)  = 0.006D0
      M1BETA(12,1)  = -0.016D0
      M1BETA(13,1)  = 0.D0
      M1BETA(14,1)  = -0.035D0
      M1BETA(15,1)  = 1.317D0
      M1BETA(16,1)  = -0.23D0
      M1BETA(17,1)  = 0.118D0
C
      M1BETA(1,2)  = 0.569D0
      M1BETA(2,2)  = 0.258D0
      M1BETA(3,2)  = 0.442D0
      M1BETA(4,2)  = 0.D0
      M1BETA(5,2)  = 0.D0
      M1BETA(6,2)  = 0.D0
      M1BETA(7,2)  = 0.153D0
      M1BETA(8,2)  = 0.002D0
      M1BETA(9,2)  = -0.001D0
      M1BETA(10,2)  = -0.286D0
      M1BETA(11,2)  = 0.020D0
      M1BETA(12,2)  = 0.D0
      M1BETA(13,2)  = -0.005D0
      M1BETA(14,2)  = -0.116D0
      M1BETA(15,2)  = 2.754D0
      M1BETA(16,2)  = -0.108D0
      M1BETA(17,2)  = 0.055D0
C
      M1BETA(1,3)  = 0.D0
      M1BETA(2,3)  = 0.D0
      M1BETA(3,3)  = 0.554D0
      M1BETA(4,3)  = -1.401D0
      M1BETA(5,3)  = -1.332D0
      M1BETA(6,3)  = -0.290D0
      M1BETA(7,3)  = 0.036D0
```

```
      M1BETA(8,3)  = -0.035D0
      M1BETA(9,3)  = 0.021D0
      M1BETA(10,3)  = 0.357D0
      M1BETA(11,3)  = -0.022D0
      M1BETA(12,3)  = -0.018D0
      M1BETA(13,3)  = 0.D0
      M1BETA(14,3)  = 0.126D0
      M1BETA(15,3)  = 1.167D0
      M1BETA(16,3)  = -0.050D0
      M1BETA(17,3)  = -0.016D0
C
      M1BETA(1,4)  = 0.D0
      M1BETA(2,4)  = 0.D0
      M1BETA(3,4)  = 1.303D0
      M1BETA(4,4)  = -0.795D0
      M1BETA(5,4)  = 0.D0
      M1BETA(6,4)  = 0.D0
      M1BETA(7,4)  = 0.010D0
      M1BETA(8,4)  = -0.047D0
      M1BETA(9,4)  = 0.046D0
      M1BETA(10,4)  = -0.326D0
      M1BETA(11,4)  = 0.220D0
      M1BETA(12,4)  = 0.D0
      M1BETA(13,4)  = 0.097D0
      M1BETA(14,4)  = -0.360D0
      M1BETA(15,4)  = 3.748D0
      M1BETA(16,4)  = -0.054D0
      M1BETA(17,4)  = 0.053D0
C
C  log of offered wage rate model estimated coefficients
C  from Table A.2
C
      M2INDX(1)  = 3
      M2INDX(2)  = 4
      M2INDX(3)  = 8
      M2INDX(4)  = 9
      M2INDX(5)  = 10
      M2INDX(6)  = 14
      M2INDX(7)  = 15
      M2INDX(8)  = 16
      M2INDX(9)  = 17
C
      M2CNST(1)  = 0.111D0
      M2CNST(2)  = 0.165D0
      M2CNST(3)  = -0.854D0
      M2CNST(4)  = 3.262D0
C
      M2BETA(1,1)  = 0.016D0
      M2BETA(2,1)  = 0.D0
      M2BETA(3,1)  = 0.D0
      M2BETA(4,1)  = 0.001D0
      M2BETA(5,1)  = -0.011D0
      M2BETA(6,1)  = 0.050D0
      M2BETA(7,1)  = 0.311D0
      M2BETA(8,1)  = -0.058D0
      M2BETA(9,1)  = 0.002D0
      M2BETA(10,1)  = 1.252D0
C
```

```
      M2BETA(1,2)  = -0.028D0
      M2BETA(2,2)  = 0.D0
      M2BETA(3,2)  = -0.001D0
      M2BETA(4,2)  = -0.002D0
      M2BETA(5,2)  = 0.006D0
      M2BETA(6,2)  = -0.054D0
      M2BETA(7,2)  = 0.533D0
      M2BETA(8,2)  = 0.018D0
      M2BETA(9,2)  = 0.003D0
      M2BETA(10,2)  = -0.494D0
C
      M2BETA(1,3)  = 0.408D0
      M2BETA(2,3)  = -0.919D0
      M2BETA(3,3)  = -0.010D0
      M2BETA(4,3)  = 0.048D0
      M2BETA(5,3)  = 0.328D0
      M2BETA(6,3)  = 0.116D0
      M2BETA(7,3)  = 0.D0
      M2BETA(8,3)  = 0.007D0
      M2BETA(9,3)  = 0.D0
      M2BETA(10,3)  = 0.807D0
C
      M2BETA(1,4)  = 2.287D0
      M2BETA(2,4)  = -2.008D0
      M2BETA(3,4)  = -0.087D0
      M2BETA(4,4)  = 0.162D0
      M2BETA(5,4)  = -2.151D0
      M2BETA(6,4)  = -1.288D0
      M2BETA(7,4)  = 0.D0
      M2BETA(8,4)  = 0.043D0
      M2BETA(9,4)  = 0.D0
      M2BETA(10,4)  = 2.508D0
C
      M2STDV(1)  = 0.50445D0
      M2STDV(2)  = 0.52223D0
      M2STDV(3)  = 0.74243D0
      M2STDV(4)  = 1.0074D0
C
C  log of annual hours of work model estimated coefficients
C  from Table A.3
C
      M3INDX(1)  = 5
      M3INDX(2)  = 6
      M3INDX(3)  = 7
      M3INDX(4)  = 8
      M3INDX(5)  = 11
      M3INDX(6)  = 12
      M3INDX(7)  = 13
C
      M3CNST(1)  = -0.193D0
      M3CNST(2)  = -0.081D0
      M3CNST(3)  = 6.714D0
      M3CNST(4)  = 7.290D0
C
      M3BETA(1,1)  = 0.D0
      M3BETA(2,1)  = 1.281D0
      M3BETA(3,1)  = -0.215D0
      M3BETA(4,1)  = 0.058D0
```

```
      M3BETA(5,1) = 0.006D0
      M3BETA(6,1) = 0.003D0
      M3BETA(7,1) = 0.002D0
      M3BETA(8,1) = -0.002D0
      M3BETA(9,1) = 0.D0
      M3BETA(10,1) = 1.115D0
C
      M3BETA(1,2) = 0.D0
      M3BETA(2,2) = -1.338D0
      M3BETA(3,2) = 0.D0
      M3BETA(4,2) = 0.D0
      M3BETA(5,2) = 0.042D0
      M3BETA(6,2) = -0.003D0
      M3BETA(7,2) = 0.014D0
      M3BETA(8,2) = 0.D0
      M3BETA(9,2) = 0.025D0
      M3BETA(10,2) = 1.578D0
C
      M3BETA(1,3) = 0.033D0
      M3BETA(2,3) = 0.D0
      M3BETA(3,3) = 0.553D0
      M3BETA(4,3) = -0.078D0
      M3BETA(5,3) = 0.050D0
      M3BETA(6,3) = -0.002D0
      M3BETA(7,3) = -0.052D0
      M3BETA(8,3) = 0.D0
      M3BETA(9,3) = 0.D0
      M3BETA(10,3) = -0.163D0
C
      M3BETA(1,4) = -0.769D0
      M3BETA(2,4) = 0.D0
      M3BETA(3,4) = 0.D0
      M3BETA(4,4) = 0.D0
      M3BETA(5,4) = 0.047D0
      M3BETA(6,4) = -0.040D0
      M3BETA(7,4) = -0.012D0
      M3BETA(8,4) = 0.D0
      M3BETA(9,4) = 0.D0
      M3BETA(10,4) = 0.337D0
C
      M3STDV(1) = 0.69995D0
      M3STDV(2) = 0.54281D0
      M3STDV(3) = 1.5211D0
      M3STDV(4) = 1.6664D0
C
      RETURN
      END
```

APPENDIX C

ESTIMATING STANDARD DEVIATION VALUES FOR THE MICROSIMULATION MODEL

A sequence of MIDAS commands was used to find the information necessary to determine the sample standard deviations to be used in the simulation model. There are four standard deviations used in the Wage Rate step, and four standard deviations used in the Hours Worked step. In each step, the four standard deviations correspond to the 2x2 classifications of the wives on age (young/old) and previous year's employment (idle/worked). For those wives who did not work in the previous year, the model dependent variables are log of wage rate and log of annual hours of work; for those wives who did work in the previous year, the model dependent variables are the differences, between the current and preceedings years, in the logs of wage rates and the logs of annual hours of work. The Nakamura paper does not provide the model standard deviation values; however, the coefficient of determination (R^2) values are provided. The model coefficients of determination and the sample standard deviations of the PSID data corresponding to the model dependent variables are used to estimate the standard deviations to use in the microsimulation model stochastic disturbance distributions.

The sequence of MIDAS commands used to code a new pair of variables for age (young/old) and employment in 1977 (idle/worked) is given in Table C.1. These commands assume that the decision unit sample is currently in the workspace; the variable numbers (on the right of the equal signs) refer to the PSID numbers. The sample standard deviations for each stratum, for the number of hours worked and for the hourly wage rate for the wives in the microunit sample that had worked in 1978 are given in the output of the two DESCRIBE commands.

Table C.1 - MIDAS Commands for Stratification

```
trans v1=v6348*v6398 lab=*
code v2=cuts(v1) points=,1, label=work78(none,some)
trans v3=v5743*v5788 lab=*
code v4=cuts(v3) points=,1, label=work77(none,some)
code v5=cuts(v5852) points=,47, label=age(young,old)
trans v11=log(v6348) label=loghrs78 case=v2:2
trans v12=v6398/v6348 label=wage78 case=same
trans v13=log(v12) label=logwag78 case=same
trans v14=log(v5743) label=loghrs77 case=same
trans v15=v11-v14 label=difloghrs case=same
trans v16=v5788/v5743 label=wage77 case=same
trans v17=log(v16) label=logwag77 case=same
trans v18=v13-v17 label=diflogwag case=same
describe bystrata v=11,13 cases=v2:2*v4:1 strata=v5
describe bystrata v=15,18 cases=v2:2*v4:2 strata=v5
```

The standard deviation for each stochastic disturbance term is found by

$$s_e = \left(\frac{\left(1 - R^2\right)\left(n - 1\right)s_y^2}{n - \left(k + 1\right)} \right)^{1/2},$$

where s_e denotes the sample standard deviation for the stochastic disturbance term , s_y denotes the sample standard deviation of the dependent variable, R^2 denotes the Nakamura model coefficient of determination, n denotes the stratum sample size, and k denotes the number of explanatory variables in the Nakamura model step. Values for s_y and n are obtained from the analysis described in Table C.1; values for R^2 and k are obtained from Nakamura and Nakamura (1985a, Tables A.1 through A.3). This analysis is performed for each stratum in the Wage Rate step and in the Hours Worked step of the microsimulation model. The sample and computed values are given in Table C.2.

Table C.2 - Estimating Standard Deviations

Step	Stratum		Sample: n	Sample: s_y	Model: k	Model: R^2	Model: s_e
wage	worked	young	375	0.50809	9	.038	0.50445
		old	221	0.51769	9	.024	0.52223
	idle	young	74	0.74980	8	.127	0.74243
		old	19	1.0598	8	.498	1.0074
hours	worked	young	375	0.71995	8	.075	0.69995
		old	221	0.59892	6	.201	0.54281
	idle	young	74	1.4863	7	.053	1.5211
		old	19	1.5753	5	.190	1.6683

APPENDIX D
INTEGRAL EVALUATIONS

This appendix presents the derivation of results used in Chapter 4 for the integration of certain functions. These results are based on three theorems given in Graybill (1969), which are presented immediately below. Here, **I** is the identity matrix, and **J** is the square matrix of ones, each with the appropriate dimensions; $\mathrm{tr}(\mathbf{M})$ denotes the trace of matrix **M**.

As stated on pages 171-172 of Graybill (1969):

Theorem 8.3.4 *Let the* $k \times k$ *matrix* **C** *be defined by*

$$\mathbf{C} = (a-b)\mathbf{I} + b\mathbf{J}.$$

The matrix **C** *has an inverse if and only if* $a \neq b$ *and* $a \neq -(k-1)b$.
If $\mathbf{C}^{-1}$ *exists, it is given by*

$$\mathbf{C}^{-1} = \frac{1}{a-b}\left[\mathbf{I} - \frac{b}{a+(k-1)b}\mathbf{J}\right]. \qquad \blacklozenge$$

As stated on page 185 of Graybill (1969), referring to the matrix **C**:

Theorem 8.4.4 *The determinant of the matrix given in Theorem 8.3.4 is equal to*

$$(a-b)^{k-1}\left[a+(k-1)b\right]. \qquad \blacklozenge$$

And, as stated on page 252 of Graybill (1969), for the evaluation of a general multiple integral:

Theorem 10.5.1 *Let* a_0 *and* b_0 *be scalar constants; let* **a** *be an* $n \times 1$ *vector of constants; let* **b** *be an* $n \times 1$ *vector of constants; let* **A** *be an* $n \times n$ *symmetric matrix of constants; let* **B** *be a positive definite matrix of constants. Then,*

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left(\mathbf{x}'\mathbf{A}\,\mathbf{x}+\mathbf{x}'\mathbf{a}+a_0\right)exp\left[-\left(\mathbf{x}'\mathbf{B}\,\mathbf{x}+\mathbf{x}'\mathbf{b}+b_0\right)\right]\,dx_1\cdots dx_n$$

$$=\frac{1}{2}\,\pi^{n/2}\,|\mathbf{B}|^{-1/2}\,exp\left[\left(\frac{1}{4}\right)\mathbf{b}'\mathbf{B}^{-1}\mathbf{b}-b_0\right]$$

$$\times\left[\mathrm{tr}\left(\mathbf{A}\mathbf{B}^{-1}\right)-\mathbf{b}'\mathbf{B}^{-1}\mathbf{a}+\frac{1}{2}\mathbf{b}'\mathbf{B}^{-1}\mathbf{A}\,\mathbf{B}^{-1}\mathbf{b}+2a_0\right],$$

where the $n\times 1$ *vector* $\mathbf{x}$ *has components* $x_1, \cdots, x_n$. ♦

The following four results are special cases of Theorem 10.5.1.

<u>Result (i)</u>

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}exp\left[-\alpha\sum_{k=1}^{m}X_k^2-\sum_{k=1}^{m}\beta_k X_k-\gamma\left(\sum_{k=1}^{m}X_k\right)^2\right]\prod_{k=1}^{m}dX_k$$

$$=\pi^{m/2}\,\alpha^{-(m-1)/2}\left(\alpha+m\gamma\right)^{-1/2}exp\left[\frac{\sum_{k=1}^{m}\left(\beta_k^2\right)}{4\alpha}-\frac{\gamma\left(\sum_{k=1}^{m}\beta_k\right)^2}{4\alpha\left(\alpha+m\gamma\right)}\right].$$

This result follows from Theorem 10.5.1, using the following definitions: $n=m$; $\mathbf{A}$ is the $m\times m$ matrix of zeros; $\mathbf{a}$ is the $m\times 1$ vector of zeros; $a_0 = 1$; $\mathbf{B} = \alpha\mathbf{I}+\gamma\mathbf{J}$; $\mathbf{b}' = \boldsymbol{\beta}' = \left(\beta_1, \ldots, \beta_m\right)'$; and $b_0 = 0$. By Theorem 8.3.4, when α and γ are restricted so that $\mathbf{B}$ is positive definite,

$$\mathbf{B}^{-1}=\frac{1}{\alpha}\left[\mathbf{I}-\frac{\gamma}{\alpha+m\gamma}\mathbf{J}\right]$$

so,

$$\mathbf{b}'\mathbf{B}^{-1}\mathbf{b} = \boldsymbol{\beta}'\left[\frac{1}{\alpha}\left(\mathbf{I} - \frac{\gamma}{\alpha + m\gamma}\mathbf{J}\right)\right]\boldsymbol{\beta}$$

$$= \frac{1}{\alpha}\left[\boldsymbol{\beta}'\mathbf{I}\boldsymbol{\beta} - \frac{\gamma}{\alpha + m\gamma}\boldsymbol{\beta}'\mathbf{J}\boldsymbol{\beta}\right]$$

$$= \frac{\sum_{k=1}^{m}\left(\beta_k^2\right)}{\alpha} - \frac{\gamma\left(\sum_{k=1}^{m}\beta_k\right)^2}{\alpha\left(\alpha + m\gamma\right)} .$$

Thus,

$$exp\left[\left(\frac{1}{4}\right)\mathbf{b}'\mathbf{B}^{-1}\mathbf{b}\right] = exp\left[\frac{\sum_{k=1}^{m}\left(\beta_k^2\right)}{4\alpha} - \frac{\gamma\left(\sum_{k=1}^{m}\beta_k\right)^2}{4\alpha\left(\alpha + m\gamma\right)}\right].$$

And by Theorem 8.4.4,

$$|\mathbf{B}| = \alpha^{m-1}\left(\alpha + m\gamma\right);$$

thus,

$$|\mathbf{B}|^{-1/2} = \alpha^{-(m-1)/2}\left(\alpha + m\gamma\right)^{-1/2}.$$

♦

Result (ii)

$$\int_{-\infty}^{+\infty} exp\left[-\alpha X^2 - \beta X\right] dX = \pi^{1/2}\,\alpha^{-1/2}\, exp\left[\beta^2\left(4\alpha\right)^{-1}\right].$$

This result follows from Theorem 10.5.1 using the following definitions: $n = 1$; $\mathbf{A} = \mathbf{a} = 0$; $a_0 = 1$; $\mathbf{B} = \alpha$; $\mathbf{b} = \beta$; and $b_0 = 0$. ♦

Result (iii)

$$\int_{-\infty}^{+\infty} X \cdot exp\left[-\alpha X^2 - \beta X\right] dX = -2^{-1}\,\pi^{1/2}\,\alpha^{-3/2}\,\beta\; exp\left[\beta^2\left(4\alpha\right)^{-1}\right].$$

This result follows from Theorem 10.5.1 using the following definitions: $n = 1$; $\mathbf{A} = 0$; $\mathbf{a} = 1$; $a_0 = 0$; $\mathbf{B} = \alpha$; $\mathbf{b} = \beta$; and $b_0 = 0$. ♦

Result (iv)

$$\int_{-\infty}^{+\infty} X^2 \cdot exp\left[-\alpha X^2 - \beta X\right] dX = 4^{-1}\,\pi^{1/2}\,\alpha^{-5/2}\left(\beta^2 + 2\alpha\right) exp\left[\beta^2\left(4\alpha\right)^{-1}\right].$$

This result follows from Theorem 10.5.1 using the following definitions: $n = 1$; $\mathbf{A} = 1$; $\mathbf{a} = 0$; $a_0 = 0$; $\mathbf{B} = \alpha$; $\mathbf{b} = \beta$; and $b_0 = 0$. ♦

APPENDIX E

STATISTICAL DENSITIES

This appendix presents definitions for the random variable distributions used in Chapter 4.

E.1 Normal Random Variable

If a random variable X has a normal distribution with paramters μ and σ^2, $-\infty < X < +\infty$, $-\infty < \mu < +\infty$, and $\sigma^2 > 0$,

$$X \sim \text{Normal}\left(\mu, \sigma^2\right),$$

then

$$f\left(X \mid \mu, \sigma\right) = \left(2\pi\sigma^2\right)^{-1/2} exp\left[-\frac{\left(X-\mu\right)^2}{2\sigma^2}\right],$$

and the random variable X has

$$\text{mean} = \mu \text{, and}$$

$$\text{variance} = \sigma^2 .$$

E.2 Inverse Gamma Random Variable

If a random variable X has an inverse gamma distribution with paramters α and β, $X > 0$, $\alpha > 0$, and $\beta > 0$,

$$X \sim \text{Inverse Gamma}\left(\alpha, \beta\right),$$

then

$$f\left(X \mid \alpha, \beta\right) = \left(\Gamma(\alpha)\, \beta^{\alpha}\right)^{-1} X^{-(\alpha - 1)}\, exp\left[\frac{-1}{X\beta}\right] \qquad \text{for } X > 0\,,$$

and the random variable X has

$$\text{mean} = \frac{1}{\beta(\alpha - 1)} \qquad \text{if } \alpha > 1\,, \text{ and}$$

$$\text{variance} = \frac{1}{\beta^2 (\alpha - 1)^2 (\alpha - 2)} \qquad \text{if } \alpha > 2\,.$$

E.3 Gamma Random Variable

If a random variable X has a gamma distribution with paramters α and β, $X > 0$, $\alpha > 0$, and $\beta > 0$,

$$X \sim \text{Gamma}\left(\alpha, \beta\right),$$

then

$$f\left(X \mid \alpha, \beta\right) = \left(\Gamma(\alpha)\, \beta^{\alpha}\right)^{-1} X^{\alpha - 1}\, exp\left[\frac{-X}{\beta}\right] \qquad \text{for } X > 0\,,$$

and the random variable X has

$$\text{mean} = \alpha\,\beta\,, \text{ and}$$

$$\text{variance} = \alpha\,\beta^2\,.$$

Note, if X has a gamma distribution, then X^{-1} has an inverse gamma distribution.

APPENDIX F

INTEGRATIONS OVER THE ROW AND COLUMN EFFECTS IN THE LIKELIHOOD FUNCTION

Let

$$Q = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g\left(R_i, C_j\right) \prod_{i=1}^{I} dR_i \prod_{j=1}^{J} dC_j ,$$

where

$$g\left(R_i, C_j\right) = exp\left[- \sum_{i=1}^{I} \frac{R_i^2}{2\ \sigma_R^2} - \sum_{j=1}^{J} \frac{C_j^2}{2\ \sigma_C^2} - \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{\left(y_{ij} - \psi - R_i - C_j\right)^2}{2\ \sigma_E^2} \right].$$

The integrations over the R_i are performed first. Completing the square and arranging terms to group those involving R_i gives

$$Q = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(exp\left[- \sum_{j=1}^{J} \frac{C_j^2}{2\ \sigma_C^2} - \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{\left(y_{ij} - \psi - C_j\right)^2}{2\ \sigma_E^2} \right] \times Q_1 \right) \prod_{j=1}^{J} dC_j ,$$

where

$$Q_1 = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} exp\left[- \sum_{i=1}^{I} \left(\frac{R_i^2}{2\ \sigma_R^2} + \sum_{j=1}^{J} \frac{R_i^2}{2\ \sigma_E^2} + \sum_{j=1}^{J} \frac{-\left(y_{ij} - \psi - C_j\right) R_i}{\sigma_E^2} \right) \right] \prod_{i=1}^{I} dR_i .$$

The multiple integrations over the R_i are independent for each i since there are no cross-product terms; the multiple integrations may be performed as a product of simple integrations. Arranging terms gives

$$Q_1 = \prod_{i=1}^{I} \left\{ \int_{-\infty}^{+\infty} exp \left[- \left(\frac{\sigma_E^2 + J\sigma_R^2}{2\,\sigma_R^2 \sigma_E^2} \right) R_i^{\,2} - \left(\frac{-\left(J\,\bar{y}_{i.} - J\,\psi - \sum_{j=1}^{J} C_j \right)}{\sigma_E^2} \right) R_i \right] dR_i \right\} .$$

The integral is evaluated using Result(ii) of Appendix D:

$$\int_{-\infty}^{+\infty} exp\left[- \alpha X^2 - \beta X \right] dX = \pi^{1/2} \alpha^{-1/2} exp\left[\beta^2 \left(4\,\alpha \right)^{-1} \right] .$$

Applying this result to the problem at hand where

$$X = R_j\,,\ \alpha = \left(\frac{\sigma_E^2 + J\sigma_R^2}{2\,\sigma_R^2 \sigma_E^2} \right), \text{ and } \beta = \left(\frac{-\left(J\,\bar{y}_{i.} - J\,\psi - \sum_{j=1}^{J} C_j \right)}{\sigma_E^2} \right)$$

gives Q_1

$$= \prod_{i=1}^{I} \left\{ \pi^{1/2} \left(\frac{\sigma_E^2 + J\sigma_R^2}{2\,\sigma_R^2 \sigma_E^2} \right)^{-1/2} exp \left[\left(\frac{-\left(J\,\bar{y}_{i.} - J\,\psi - \sum_{j=1}^{J} C_j \right)}{\sigma_E^2} \right)^2 \left(\frac{4\left(\sigma_E^2 + J\sigma_R^2 \right)}{2\,\sigma_R^2 \sigma_E^2} \right)^{-1} \right] \right\}$$

$$= \pi^{1/2} \left(\frac{\sigma_E^2 + J\sigma_R^2}{2\,\sigma_R^2 \sigma_E^2} \right)^{-1/2} exp \left[\sum_{i=1}^{I} \frac{\sigma_R^2 \left(J\,\bar{y}_{i.} - J\,\psi - \sum_{j=1}^{J} C_j \right)^2}{2\,\sigma_E^2 \left(\sigma_E^2 + J\sigma_R^2 \right)} \right] .$$

Substituting the evaluated integral into the expression for Q gives

$$Q = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} Q_2 \prod_{j=1}^{J} dC_j$$

where Q_2

$$= exp\left[- \sum_{j=1}^{J} \frac{C_j^{\,2}}{2\,\sigma_C^2} - \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{\left(y_{ij} - \psi - C_j\right)^2}{2\,\sigma_E^2} \right]$$

$$\times\ \pi^{1/2} \left(\frac{\sigma_E^2 + J\sigma_R^2}{2\,\sigma_R^2\,\sigma_E^2} \right)^{-1/2} exp\left[\sum_{i=1}^{I} \frac{\sigma_R^2 \left(J\,\bar{y}_{i.} - J\,\psi - \sum_{j=1}^{J} C_j \right)^2}{2\,\sigma_E^2 \left(\sigma_E^2 + J\sigma_R^2 \right)} \right]$$

$$= \left(2\,\pi\,\sigma_R^2\,\sigma_E^2 \right)^{1/2} \left(\sigma_E^2 + J\sigma_R^2 \right)^{-1/2}$$

$$\times\ exp\left[- \frac{\sum_{j=1}^{J} \left(C_j^{\,2} \right)}{2\,\sigma_C^2} - \frac{\sum_{i=1}^{I} \sum_{j=1}^{J} \left(+ y_{ij}^{\,2} - 2\,y_{ij}\,\psi - 2\,y_{ij}\,C_j + \psi^2 + 2\,\psi\,C_j + C_j^{\,2} \right)}{2\,\sigma_E^2} \right]$$

$$\times\ exp\left[+ \frac{\sigma_R^2 \sum_{i=1}^{I} \left[+ \left(J\,\bar{y}_{i.} \right)^2 - 2\,J^2\,\bar{y}_{i.}\,\psi - 2\,J\,\bar{y}_{i.} \sum_{j=1}^{J} C_j \right]}{2\,\sigma_E^2 \left(\sigma_E^2 + J\sigma_R^2 \right)} \right]$$

$$\times\ exp\left[+ \frac{\sigma_R^2 \sum_{i=1}^{I} \left[+ \left(J\,\psi \right)^2 + 2\,J\,\psi \sum_{j=1}^{J} C_j + \left(\sum_{j=1}^{J} C_j \right)^2 \right]}{2\,\sigma_E^2 \left(\sigma_E^2 + J\sigma_R^2 \right)} \right]$$

$$= \left(2\pi\sigma_R^2\sigma_E^2\right)^{1/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{-1/2} exp\left[-\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}^{2}\right)}{2\,\sigma_E^2} + \frac{J^2\sigma_R^2\sum_{i=1}^{I}\left(\bar{y}_{i.}^{2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]$$

$$\times exp\left[+\frac{IJ\,\bar{y}_{..}\,\psi}{\sigma_E^2} - \frac{IJ\,\psi^2}{2\,\sigma_E^2} - \frac{IJ^2\,\sigma_R^2\,\bar{y}_{..}\,\psi}{\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)} + \frac{IJ^2\,\sigma_R^2\,\psi^2}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]$$

$$\times exp\left[-\frac{\sum_{j=1}^{J}\left(C_j^{2}\right)}{2\,\sigma_C^2} - \frac{I\sum_{j=1}^{J}\left(C_j^{2}\right)}{2\,\sigma_E^2} + \frac{I\sigma_R^2\left(\sum_{j=1}^{J} C_j\right)^2}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]$$

$$\times exp\left[+\frac{I\sum_{j=1}^{J}\left(\bar{y}_{.j}C_j\right)}{\sigma_E^2} - \frac{I\,\psi\sum_{j=1}^{J} C_j}{\sigma_E^2} - \frac{IJ\sigma_R^2\,\bar{y}_{..}\sum_{j=1}^{J} C_j}{\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)} + \frac{IJ\sigma_R^2\,\psi\sum_{j=1}^{J} C_j}{\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]$$

$$= \left(2\pi\sigma_R^2\sigma_E^2\right)^{1/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{-1/2} exp\left[-\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}^{2}\right)}{2\,\sigma_E^2} + \frac{J^2\sigma_R^2\sum_{i=1}^{I}\left(\bar{y}_{i.}^{2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]$$

$$\times\, exp\left[+\frac{IJ\,\bar{y}_{..}\,\psi}{\sigma_E^2 + J\sigma_R^2} - \frac{IJ\,\psi^2}{2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]$$

$$\times\, exp\left[-\frac{\sigma_E^2 + I\sigma_C^2}{2\,\sigma_C^2\sigma_E^2}\sum_{j=1}^{J}\left(C_j^{2}\right) + \frac{I\sigma_R^2}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)}\left(\sum_{j=1}^{J} C_j\right)^2\right]$$

$$\times\, exp\left[-\sum_{j=1}^{J}\left(\frac{IJ\sigma_R^2\,\bar{y}_{..}}{\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{I\bar{y}_{j}}{\sigma_E^2} + \frac{I\psi}{\left(\sigma_E^2 + J\sigma_R^2\right)}\right)C_j\right].$$

Removing from the integrals all terms not involving C_j gives

$$Q = \left(2\pi\sigma_R^2\sigma_E^2\right)^{1/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{-1/2} exp\left[-\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}^{2}\right)}{2\,\sigma_E^2} + \frac{J^2\sigma_R^2\sum_{i=1}^{I}\left(\bar{y}_{i.}^{2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]$$

$$\times\, exp\left[+\frac{IJ\,\bar{y}_{..}\,\psi}{\sigma_E^2 + J\sigma_R^2} - \frac{IJ\,\psi^2}{2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} Q_3 \prod_{j=1}^{J} dC_j\,,$$

where

$$Q_3 = exp\left[- \left(\frac{\sigma_E^2 + I\sigma_C^2}{2\,\sigma_C^2 \sigma_E^2} \right) \left(\sum_{j=1}^{J} C_j^{\,2} \right) - \sum_{j=1}^{J} \left(\frac{IJ\sigma_R^2 \bar{y}_{..}}{\sigma_E^2 \left(\sigma_E^2 + J\sigma_R^2 \right)} - \frac{I\bar{y}_j}{\sigma_E^2} + \frac{I\,\psi}{\sigma_E^2 + J\sigma_R^2} \right) C_j \right]$$

$$\times\, exp\left[- \left(\frac{-I\sigma_R^2}{2\,\sigma_E^2 \left(\sigma_E^2 + J\sigma_R^2 \right)} \right) \left(\sum_{j=1}^{J} C_j \right)^2 \right].$$

The integrals are evaluated using Result(i) of Appendix D:

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} exp\left[-\alpha \sum_{k=1}^{m} X_k^{\,2} - \sum_{k=1}^{m} \beta_k X_k - \gamma \left(\sum_{k=1}^{m} X_k \right)^2 \right] \prod_{k=1}^{m} dX_k$$

$$= \pi^{m/2} \alpha^{-(m-1)/2} \left(\alpha + m\gamma \right)^{-1/2} exp\left[\frac{\sum_{k=1}^{m} \left(\beta_k^{\,2} \right)}{4\alpha} - \frac{\gamma \left(\sum_{k=1}^{m} \beta_k \right)^2}{4\alpha \left(\alpha + m\gamma \right)} \right].$$

Applying this result to the problem at hand, where $k = j$, $X_k = C_j$, $m = J$,

$$\alpha = \left(\frac{\sigma_E^2 + I\sigma_C^2}{2\,\sigma_C^2 \sigma_E^2} \right), \quad \beta_k = \left(\frac{IJ\sigma_R^2 \bar{y}_{..}}{\sigma_E^2 \left(\sigma_E^2 + J\sigma_R^2 \right)} - \frac{I\bar{y}_j}{\sigma_E^2} + \frac{I\,\psi}{\sigma_E^2 + J\sigma_R^2} \right), \text{ and}$$

$$\gamma = \left(\frac{-I\sigma_R^2}{2\,\sigma_E^2 \left(\sigma_E^2 + J\sigma_R^2 \right)} \right) \text{gives}$$

$$\left(\alpha + m\gamma\right) = \left(\frac{\sigma_E^2 + I\sigma_C^2}{2\,\sigma_C^2\sigma_E^2}\right) + J\left(\frac{-\,I\sigma_R^2}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)}\right)$$

$$= \frac{\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2\right) - IJ\sigma_C^2\sigma_R^2}{2\,\sigma_C^2\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)}$$

$$= \frac{\left(\sigma_E^2\right)^2 + J\sigma_E^2\sigma_R^2 + I\sigma_E^2\sigma_C^2 + IJ\sigma_C^2\sigma_R^2 - IJ\sigma_C^2\sigma_R^2}{2\,\sigma_C^2\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)}$$

$$= \frac{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}{2\,\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2\right)}$$

and,

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} Q_3 \prod_{j=1}^{J} dC_j$$

$$= \pi^{J/2} \left(\frac{\sigma_E^2 + I\sigma_C^2}{2\,\sigma_C^2 \sigma_E^2} \right)^{-(J-1)/2} \left(\frac{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}{2\,\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2\right)} \right)^{-1/2}$$

$$\times\, exp\left[+ \frac{\sum_{j=1}^{J} \left(\frac{IJ\sigma_R^2 \bar{y}_{.}}{\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{I\,\bar{y}_j}{\sigma_E^2} + \frac{I\,\psi}{\sigma_E^2 + J\sigma_R^2} \right)^2}{4\left(\frac{\sigma_E^2 + I\sigma_C^2}{2\,\sigma_C^2 \sigma_E^2} \right)} \right]$$

$$\times\, exp\left[- \frac{\left(\frac{-\,I\sigma_R^2}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)} \right) \left[\sum_{j=1}^{J} \left(\frac{IJ\sigma_R^2 \bar{y}_{.}}{\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{I\,\bar{y}_j}{\sigma_E^2} + \frac{I\,\psi}{\sigma_E^2 + J\sigma_R^2} \right) \right]^2}{4\left(\frac{\sigma_E^2 + I\sigma_C^2}{2\,\sigma_C^2 \sigma_E^2} \right)\left(\frac{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}{2\,\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2\right)} \right)} \right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{1/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}$$

$$\times \exp\left[+\frac{\sum_{j=1}^{J}\left(\frac{IJ\sigma_R^2\bar{y}_{..}}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)}\right)^2}{2\left(\frac{\sigma_E^2+I\sigma_C^2}{\sigma_C^2\sigma_E^2}\right)}-\frac{\sum_{j=1}^{J}\left(\frac{IJ\sigma_R^2\bar{y}_{..}}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)}\right)\left(\frac{I\bar{y}_{.j}}{\sigma_E^2}\right)}{\frac{\sigma_E^2+I\sigma_C^2}{\sigma_C^2\sigma_E^2}}\right]$$

$$\times \exp\left[+\frac{\sum_{j=1}^{J}\left(\frac{IJ\sigma_R^2\bar{y}_{..}}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)}\right)\left(\frac{I\psi}{\sigma_E^2+J\sigma_R^2}\right)}{\frac{\sigma_E^2+I\sigma_C^2}{\sigma_C^2\sigma_E^2}}+\frac{\sum_{j=1}^{J}\left(-\frac{I\bar{y}_{.j}}{\sigma_E^2}\right)^2}{2\left(\frac{\sigma_E^2+I\sigma_C^2}{\sigma_C^2\sigma_E^2}\right)}\right]$$

$$\times \exp\left[-\frac{\sum_{j=1}^{J}\left(\frac{I\bar{y}_{.j}}{\sigma_E^2}\right)\left(\frac{I\psi}{\sigma_E^2+J\sigma_R^2}\right)}{\frac{\sigma_E^2+I\sigma_C^2}{\sigma_C^2\sigma_E^2}}+\frac{\sum_{j=1}^{J}\left(\frac{I\psi}{\sigma_E^2+J\sigma_R^2}\right)^2}{2\left(\frac{\sigma_E^2+I\sigma_C^2}{\sigma_C^2\sigma_E^2}\right)}\right]$$

$$\times \exp\left[+\frac{I\sigma_R^2\left(\sigma_C^2\right)^2\left(\frac{IJ^2\sigma_R^2\bar{y}_{..}}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)}-\frac{IJ\,\bar{y}_{..}}{\sigma_E^2}+\frac{IJ\,\psi}{\sigma_E^2+J\sigma_R^2}\right)^2}{2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{1/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times\, exp\left[+\frac{I^2 J^3 \left(\sigma_R^2\right)^2 \sigma_C^2 \bar{y}_{..}^{\,2}}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{I^2 J^2 \sigma_R^2 \sigma_C^2 \bar{y}_{..}^{\,2}}{\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{I^2 J^2 \sigma_R^2 \sigma_C^2 \bar{y}_{..}\psi}{\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)} + \frac{I^2 \sigma_C^2 \sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{I^2 J\sigma_C^2 \psi\,\bar{y}_{..}}{\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + I\sigma_C^2\right)} + \frac{I^2 J\sigma_C^2 \sigma_E^2 \psi^2}{2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{I\sigma_R^2\left(\sigma_C^2\right)^2\left(+\frac{IJ^2\sigma_R^2\bar{y}_{..}}{\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{IJ\,\bar{y}_{..}\left(\sigma_E^2 + J\sigma_R^2\right)}{\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)} + \frac{IJ\,\psi}{\sigma_E^2 + J\sigma_R^2}\right)^2}{2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{1/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}$$

$$\times\, exp\left[+\frac{I^2J^3\left(\sigma_R^2\right)^2\sigma_C^2\bar{y}_{..}^{\,2}}{2\,\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)}-\frac{I^2J^2\sigma_R^2\sigma_C^2\bar{y}_{..}^{\,2}}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{I^2J^2\sigma_R^2\sigma_C^2\bar{y}_{..}\psi}{\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)}+\frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right)}{2\,\sigma_E^2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{I^2J\sigma_C^2\psi\,\bar{y}_{..}}{\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)}+\frac{I^2J\sigma_C^2\sigma_E^2\psi^{\,2}}{2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{I\sigma_R^2\left(\sigma_C^2\right)^2\left(-\frac{IJ\,\bar{y}_{..}\sigma_E^2}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)}+\frac{IJ\,\psi}{\sigma_E^2+J\sigma_R^2}\right)^2}{2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{1/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times\, exp\left[+\frac{I^2 J^3\left(\sigma_R^2\right)^2\sigma_C^2\bar{y}_{..}^{\,2}}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{I^2 J^2\sigma_R^2\sigma_C^2\bar{y}_{..}^{\,2}}{\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{I^2 J^2\sigma_R^2\sigma_C^2\bar{y}_{..}\psi}{\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)} + \frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{I^2 J\,\sigma_C^2\psi\,\bar{y}_{..}}{\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + I\sigma_C^2\right)} + \frac{I^2 J\,\sigma_C^2\sigma_E^2\psi^2}{2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{I\,\sigma_R^2\left(\sigma_C^2\right)^2\left(-\frac{IJ\,\bar{y}_{..}}{\sigma_E^2 + J\sigma_R^2} + \frac{IJ\,\psi}{\sigma_E^2 + J\sigma_R^2}\right)^2}{2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$= \left(2\,\pi\,\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{1/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times\, exp\left[+ \frac{I^2 J^3 \left(\sigma_R^2\right)^2 \sigma_C^2 \bar{y}_{..}^{\,2}}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{I^2 J^2 \sigma_R^2 \sigma_C^2 \bar{y}_{..}^{\,2}}{\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+ \frac{I^2 J^2 \sigma_R^2 \sigma_C^2 \bar{y}_{..}\psi}{\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)} + \frac{I^2 \sigma_C^2 \sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[- \frac{I^2 J\, \sigma_C^2 \psi\, \bar{y}_{..}}{\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + I\sigma_C^2\right)} + \frac{I^2 J\, \sigma_C^2 \sigma_E^2 \psi^2}{2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+ \frac{I^3 J^2 \sigma_R^2 \left(\sigma_C^2\right)^2\left(\psi^2 - 2\psi\,\bar{y}_{..} + \bar{y}_{..}^{\,2}\right)}{2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{1/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[+\frac{I^3J^2\sigma_R^2\left(\sigma_C^2\right)^2\psi^2}{2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^2J\sigma_C^2\sigma_E^2\psi^2}{2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^2J^2\sigma_R^2\sigma_C^2\bar{y}_{..}\psi}{\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)}-\frac{I^2J\sigma_C^2\psi\bar{y}_{..}}{\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{I^3J^2\sigma_R^2\left(\sigma_C^2\right)^2\psi\bar{y}_{..}}{\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{j}^{\,2}\right)}{2\sigma_E^2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^2J^3\left(\sigma_R^2\right)^2\sigma_C^2\bar{y}_{..}^{\,2}}{2\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)}-\frac{I^2J^2\sigma_R^2\sigma_C^2\bar{y}_{..}^{\,2}}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^3J^2\sigma_R^2\left(\sigma_C^2\right)^2\bar{y}_{..}^{\,2}}{2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{1/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[+\frac{I^3 J^2 \sigma_R^2\left(\sigma_C^2\right)^2\psi^2 + I^2 J \sigma_C^2\sigma_E^2\psi^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}{2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^2 J^2 \sigma_R^2\sigma_C^2\bar{y}_{..}\psi\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{I^2 J \sigma_C^2\bar{y}_{..}\psi\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{I^3 J^2 \sigma_R^2\left(\sigma_C^2\right)^2\bar{y}_{..}\psi -}{\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} + \frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\;2}\right)}{2\sigma_E^2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^2 J^3\left(\sigma_R^2\right)^2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\bar{y}_{..}^{\;2}}{2\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{2I^2 J^2 \sigma_R^2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\bar{y}_{..}^{\;2}}{2\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^3 J^2 \sigma_R^2\left(\sigma_C^2\right)^2\sigma_E^2\bar{y}_{..}^{\;2}}{2\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{1/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}$$

$$\times\, exp\left[+\frac{I^2 J\sigma_C^2\psi^2\left[IJ\sigma_R^2\sigma_C^2+\left(\sigma_E^2\right)^2+J\sigma_R^2\sigma_E^2+I\sigma_C^2\sigma_E^2\right]}{2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{I^2 J\sigma_C^2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)\bar{y}_{..}\psi\left(J\sigma_R^2-\sigma_E^2-J\sigma_R^2\right)}{\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{I^3 J^2\sigma_R^2\left(\sigma_C^2\right)^2\bar{y}_{..}\psi}{\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right)}{2\,\sigma_E^2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{I^2J^2\sigma_R^2\sigma_C^2\bar{y}_{..}^{\,2}\left[-J\sigma_R^2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)-I\sigma_C^2\sigma_E^2\right]}{2\,\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{I^2J^2\sigma_R^2\sigma_C^2\bar{y}_{..}^{\,2}\left[+{}^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)\right]}{2\,\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{1/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[+\frac{I^2 J\sigma_C^2\psi^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)}{2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{I^2 J\sigma_C^2\sigma_E^2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)\bar{y}_{..}\psi+I^3J^2\sigma_R^2\left(\sigma_C^2\right)^2\bar{y}_{..}\psi}{\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{.j}{}^2\right)}{2\sigma_E^2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{I^2J^2\sigma_R^2\sigma_C^2\bar{y}_{..}{}^2\begin{pmatrix}-J\sigma_R^2\sigma_E^2-\left(J\sigma_R^2\right)^2-IJ\sigma_R^2\sigma_C^2-I\sigma_C^2\sigma_E^2\\+2\left(\sigma_E^2\right)^2+4J\sigma_R^2\sigma_E^2+2I\sigma_C^2\sigma_E^2\\+2\left(J\sigma_R^2\right)^2+2IJ\sigma_R^2\sigma_C^2\end{pmatrix}}{2\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{1/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[+\frac{I^2 J\sigma_C^2\psi^2}{2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{I^2 J\sigma_C^2\bar{y}_{..}\psi\left[\left(\sigma_E^2\right)^2+J\sigma_R^2\sigma_E^2+I\sigma_C^2\sigma_E^2+IJ\sigma_R^2\sigma_C^2\right]}{\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right)}{2\sigma_E^2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{I^2 J^2\sigma_R^2\sigma_C^2\bar{y}_{..}^{\,2}\left[2\left(\sigma_E^2\right)^2+3J\sigma_R^2\sigma_E^2+I\sigma_C^2\sigma_E^2+\left(J\sigma_R^2\right)^2+IJ\sigma_R^2\sigma_C^2\right]}{2\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{1/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[+ \frac{I^2 J\sigma_C^2 \psi^2}{2\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[- \frac{I^2 J\sigma_C^2 \bar{y}_{..}\psi\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + I\sigma_C^2\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[+ \frac{I^2 \sigma_C^2 \sum_{j=1}^{J}\left(\bar{y}_{j}^{\;2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[- \frac{I^2 J^2 \sigma_R^2 \sigma_C^2 \bar{y}_{..}^{\;2}\left(\sigma_E^2 + J\sigma_R^2\right)\left(2\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)^2\left(\sigma_E^2 + I\sigma_C^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{1/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[+\frac{I^2 J\,\sigma_C^2\psi^2}{2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}-\frac{I^2 J\,\sigma_C^2\bar{y}_{..}\psi}{\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right)}{2\,\sigma_E^2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{I^2 J^2\sigma_R^2\sigma_C^2\bar{y}_{..}^{\,2}\left(2\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}{2\,\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{1/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[+\frac{I^2 J\,\sigma_C^2\psi^2}{2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}-\frac{I^2 J\,\sigma_C^2\bar{y}_{..}\psi}{\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right)}{2\,\sigma_E^2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{IJ\,\bar{y}_{..}^{\,2}}{2}\left(\frac{IJ\,\sigma_R^2\sigma_C^2\left(2\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right)\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{1/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}$$

$$\times\, exp\left[+\frac{I^2 J\,\sigma_C^2\psi^2}{2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}-\frac{I^2 J\,\sigma_C^2\bar{y}_{..}\psi}{\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right)}{2\,\sigma_E^2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{IJ\,\bar{y}_{..}^{\,2}}{2}\left(\frac{\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right)\right]$$

$$\times\, exp\left[-\frac{IJ\,\bar{y}_{..}^{\,2}}{2}\left(\frac{-\,\sigma_E^2\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right)\right]$$

$$\times\, exp\left[-\frac{IJ\,\bar{y}_{..}^{\,2}}{2}\left(\frac{-\,\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right)\right]$$

$$\times\, exp\left[-\frac{IJ\,\bar{y}_{..}^{\,2}}{2}\left(\frac{+\,\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)}{\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+I\sigma_C^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right)\right]$$

$$= \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{1/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times\, exp\left[+\frac{I^2 J\,\sigma_C^2\psi^2}{2\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} - \frac{I^2 J\,\sigma_C^2\bar{y}_{..}\psi}{\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{IJ\,\bar{y}_{..}^{\,2}}{2\,\sigma_E^2} + \frac{IJ\,\bar{y}_{..}^{\,2}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} + \frac{IJ\,\bar{y}_{..}^{\,2}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{IJ\,\bar{y}_{..}^{\,2}}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right].$$

Substituting the evaluated integrals into the expression for Q gives

$$Q = \left(2\pi\sigma_R^2\sigma_E^2\right)^{I/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{-I/2} exp\left[-\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}^{\,2}\right)}{2\,\sigma_E^2} + \frac{J^2\sigma_R^2\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\,2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]$$

$$\times\, exp\left[+\frac{IJ\,\bar{y}_{..}\psi}{\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{IJ\,\psi^2}{2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]$$

$$\times \left(2\pi\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(J-1)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{1/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times\, exp\left[+\frac{I^2 J\,\sigma_C^2\psi^2}{2\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} - \frac{I^2 J\,\sigma_C^2\bar{y}_{..}\psi}{\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{j}^{\,2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{IJ\,\bar{y}_{..}^{\,2}}{2\,\sigma_E^2} + \frac{IJ\,\bar{y}_{..}^{\,2}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} + \frac{IJ\,\bar{y}_{..}^{\,2}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{IJ\,\bar{y}_{..}^{\,2}}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\right)^{(I+J)/2}\left(\sigma_R^2\right)^{I/2}\left(\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(I+J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I-1)/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}\ exp\left[-\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}^{\ 2}\right)}{2\,\sigma_E^2}\right]$$

$$\times exp\left[-\frac{IJ\,\bar{y}_{..}^{\ 2}}{2\,\sigma_E^2}+\frac{IJ\,\bar{y}_{..}^{\ 2}}{2\left(\sigma_E^2+J\sigma_R^2\right)}+\frac{IJ\,\bar{y}_{..}^{\ 2}}{2\left(\sigma_E^2+I\sigma_C^2\right)}-\frac{IJ\,\bar{y}_{..}^{\ 2}}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{IJ\,\bar{y}_{..}\psi}{\left(\sigma_E^2+J\sigma_R^2\right)}-\frac{I^2J\sigma_C^2\bar{y}_{..}\psi}{\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{IJ\,\psi^2}{2\left(\sigma_E^2+J\sigma_R^2\right)}+\frac{I^2J\sigma_C^2\psi^2}{2\left(\sigma_E^2+J\sigma_R^2\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[+\frac{J^2\sigma_R^2\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\ 2}\right)}{2\,\sigma_E^2\left(\sigma_E^2+J\sigma_R^2\right)}+\frac{I^2\sigma_C^2\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\ 2}\right)}{2\,\sigma_E^2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\right)^{(I+J)/2}\left(\sigma_R^2\right)^{1/2}\left(\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(I+J-1)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{-(I-1)/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} \; exp\left[- \frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}^{\;2}\right)}{2\,\sigma_E^2} \right]$$

$$\times exp\left[- \frac{IJ\,\bar{y}_{..}^{\;2}}{2\,\sigma_E^2} + \frac{IJ\,\bar{y}_{..}^{\;2}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} + \frac{IJ\,\bar{y}_{..}^{\;2}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{IJ\,\bar{y}_{..}^{\;2}}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right]$$

$$\times exp\left[+ \frac{IJ\,\bar{y}_{..}\,\psi\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) - I^2 J\sigma_C^2\bar{y}_{..}\,\psi}{\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right]$$

$$\times exp\left[- \frac{IJ\,\psi^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) - I^2 J\sigma_C^2\psi^2}{2\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right]$$

$$\times exp\left[+ \frac{J\left[\left(\sigma_E^2 + J\sigma_R^2\right) - \sigma_E^2\right]\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\;2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)} + \frac{I\left[\left(\sigma_E^2 + I\sigma_C^2\right) - \sigma_E^2\right]\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\;2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + I\sigma_C^2\right)} \right]$$

$$= \left(2\pi\right)^{(I+J)/2} \left(\sigma_R^2\right)^{I/2} \left(\sigma_C^2\right)^{J/2} \left(\sigma_E^2\right)^{(I+J-1)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I-1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} exp\left[- \frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}^{\;2}\right)}{2\,\sigma_E^2} \right]$$

$$\times exp\left[- \frac{IJ\,\bar{y}_{..}^{\;2}}{2\,\sigma_E^2} + \frac{IJ\,\bar{y}_{..}^{\;2}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} + \frac{IJ\,\bar{y}_{..}^{\;2}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{IJ\,\bar{y}_{..}^{\;2}}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right]$$

$$\times exp\left[+ \frac{IJ\,\bar{y}_{..}\,\psi\left(\sigma_E^2 + J\sigma_R^2\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} - \frac{IJ\,\psi^2\left(\sigma_E^2 + J\sigma_R^2\right)}{2\left(\sigma_E^2 + J\sigma_R^2\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right]$$

$$\times exp\left[+ \frac{J\left(\sigma_E^2 + J\sigma_R^2\right)\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\;2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{J\,\sigma_E^2\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\;2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + J\sigma_R^2\right)} \right]$$

$$\times exp\left[+ \frac{I\left(\sigma_E^2 + I\sigma_C^2\right)\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\;2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{I\,\sigma_E^2\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\;2}\right)}{2\,\sigma_E^2\left(\sigma_E^2 + I\sigma_C^2\right)} \right]$$

$$= \left(2\pi\right)^{(I+J)/2} \left(\sigma_R^2\right)^{I/2} \left(\sigma_C^2\right)^{J/2} \left(\sigma_E^2\right)^{(I+J-1)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I-1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} \; exp\left[- \frac{\sum_{i=1}^{I} \sum_{j=1}^{J} \left(y_{ij}^{\;2}\right)}{2\,\sigma_E^2} \right]$$

$$\times\, exp\left[- \frac{IJ\,\bar{y}_{..}^{\;2}}{2\,\sigma_E^2} + \frac{IJ\,\bar{y}_{..}^{\;2}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} + \frac{IJ\,\bar{y}_{..}^{\;2}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{IJ\,\bar{y}_{..}^{\;2}}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right]$$

$$\times\, exp\left[+ \frac{IJ\,\bar{y}_{..}\,\psi}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} - \frac{IJ\,\psi^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right]$$

$$\times\, exp\left[+ \frac{J\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\;2}\right)}{2\,\sigma_E^2} - \frac{J\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\;2}\right)}{2\left(\sigma_E^2 + J\sigma_R^2\right)} + \frac{I\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\;2}\right)}{2\,\sigma_E^2} - \frac{I\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\;2}\right)}{2\left(\sigma_E^2 + I\sigma_C^2\right)} \right]$$

$$= \left(2\pi\right)^{(I+J)/2}\left(\sigma_R^2\right)^{I/2}\left(\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(I+J-1)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{-(I-1)/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[-\frac{IJ\,\bar{y}_{..}^{\,2} - 2\,IJ\,\bar{y}_{..}\,\psi + IJ\,\psi^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}^{\,2}\right) - J\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\,2}\right) - I\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right) + IJ\,\bar{y}_{..}^{\,2}}{2\,\sigma_E^2}\right]$$

$$\times exp\left[-\frac{J\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\,2}\right) - IJ\,\bar{y}_{..}^{\,2}}{2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]$$

$$\times exp\left[-\frac{I\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\,2}\right) - IJ\,\bar{y}_{..}^{\,2}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\right)^{(I+J)/2}\left(\sigma_R^2\right)^{1/2}\left(\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(I+J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I-1)/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2} exp\left[-\frac{IJ\left(\bar{y}_{..}-\psi\right)^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{\left(\begin{array}{l}+\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}^{\;2}\right)-2J\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\;2}\right)+J\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\;2}\right)\\ -2I\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\;2}\right)+I\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\;2}\right)+IJ\,\bar{y}_{..}^{\;2}\end{array}\right)}{2\,\sigma_E^2}\right]$$

$$\times exp\left[-\frac{J\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\;2}\right)-2\,IJ\,\bar{y}_{..}^{\;2}+IJ\,\bar{y}_{..}^{\;2}}{2\left(\sigma_E^2+J\sigma_R^2\right)}\right]$$

$$\times exp\left[-\frac{I\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\;2}\right)-2\,IJ\,\bar{y}_{..}^{\;2}+IJ\,\bar{y}_{..}^{\;2}}{2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\right)^{(I+J)/2}\left(\sigma_R^2\right)^{1/2}\left(\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(I+J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I-1)/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2} exp\left[-\frac{IJ\left(\bar{y}_{..}-\psi\right)^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{\left(\begin{array}{l}+\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}^{2}\right)-2\sum_{i=1}^{I}\left(J\,\bar{y}_{i}\right)\bar{y}_{i.}+\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\bar{y}_{i.}^{2}\right)\\-2\sum_{j=1}^{J}\left(I\,\bar{y}_{j}\right)\bar{y}_{.j}+\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\bar{y}_{.j}^{2}\right)+\sum_{i=1}^{I}\sum_{j=1}^{J}\bar{y}_{..}^{2}\\-2IJ\,\bar{y}_{..}^{2}-2IJ\,\bar{y}_{..}^{2}+2IJ\,\bar{y}_{..}^{2}+2IJ\,\bar{y}_{..}^{2}\end{array}\right)}{2\,\sigma_E^2}\right]$$

$$\times exp\left[-\frac{J\sum_{i=1}^{I}\left(\bar{y}_{i.}^{2}\right)-2J\,\bar{y}_{..}I\,\bar{y}_{..}+IJ\,\bar{y}_{..}^{2}}{2\left(\sigma_E^2+J\sigma_R^2\right)}\right]$$

$$\times exp\left[-\frac{I\sum_{j=1}^{J}\left(\bar{y}_{.j}^{2}\right)-2I\,\bar{y}_{..}J\,\bar{y}_{..}+IJ\,\bar{y}_{..}^{2}}{2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\right)^{(I+J)/2}\left(\sigma_R^2\right)^{I/2}\left(\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(I+J-1)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{-(I-1)/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} exp\left[-\frac{IJ\left(\bar{y}_{..} - \psi\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\begin{array}{c} y_{ij}^{\ 2} - 2\,y_{ij}\bar{y}_{i.} + \bar{y}_{i.}^{\ 2} - 2\,y_{ij}\bar{y}_{.j} + \bar{y}_{.j}^{\ 2} + \bar{y}_{..}^{\ 2} \\ -\,2\,\bar{y}_{i.}\bar{y}_{..} - 2\,\bar{y}_{.j}\bar{y}_{..} + 2\,y_{ij}\bar{y}_{..} + 2\,\bar{y}_{i.}\bar{y}_{.j} \end{array}\right)}{2\,\sigma_E^2}\right]$$

$$\times exp\left[-\frac{J\sum_{i=1}^{I}\left(\bar{y}_{i.}^{\ 2}\right) - 2J\,\bar{y}_{..}\sum_{i=1}^{I}\left(\bar{y}_{i.}\right) + J\sum_{i=1}^{I}\bar{y}_{..}^{\ 2}}{2\left(\sigma_E^2 + J\sigma_R^2\right)}\right]$$

$$\times exp\left[-\frac{I\sum_{j=1}^{J}\left(\bar{y}_{.j}^{\ 2}\right) - 2I\,\bar{y}_{..}\sum_{j=1}^{J}\left(\bar{y}_{.j}\right) + I\sum_{j=1}^{J}\bar{y}_{..}^{\ 2}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\right)^{(I+J)/2}\left(\sigma_R^2\right)^{I/2}\left(\sigma_C^2\right)^{J/2}\left(\sigma_E^2\right)^{(I+J-1)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I-1)/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2} exp\left[-\frac{IJ\left(\bar{y}_{..}-\psi\right)^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\begin{array}{c}+y_{ij}^2-2y_{ij}\bar{y}_{i.}-2y_{ij}\bar{y}_{.j}+2y_{ij}\bar{y}_{..}+\bar{y}_{i.}^2\\ +2\bar{y}_{i.}\bar{y}_{.j}-2\bar{y}_{i.}\bar{y}_{..}+\bar{y}_{.j}^2-2\bar{y}_{.j}\bar{y}_{..}+\bar{y}_{..}^2\end{array}\right)}{2\sigma_E^2}\right]$$

$$\times exp\left[-\frac{J\sum_{i=1}^{I}\left(\bar{y}_{i.}^2-{}^2\bar{y}_{i.}\bar{y}_{..}+\bar{y}_{..}^2\right)}{2\left(\sigma_E^2+J\sigma_R^2\right)}-\frac{I\sum_{j=1}^{J}\left(\bar{y}_{.j}^2-{}^2\bar{y}_{.j}\bar{y}_{..}+\bar{y}_{..}^2\right)}{2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$= \left(2\pi\right)^{(I+J)/2} \left(\sigma_R^2\right)^{1/2} \left(\sigma_C^2\right)^{J/2} \left(\sigma_E^2\right)^{(I+J-1)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I-1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times\ exp\left[-\frac{IJ\left(\bar{y}_{..} - \psi\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} - \frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}\right)^2}{2\,\sigma_E^2} \right]$$

$$\times\ exp\left[-\frac{J\sum_{i=1}^{I}\left(\bar{y}_{i.} - \bar{y}_{..}\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{I\sum_{j=1}^{J}\left(\bar{y}_{.j} - \bar{y}_{..}\right)^2}{2\left(\sigma_E^2 + I\sigma_C^2\right)} \right]$$

$$= \left(2\pi\right)^{(I+J)/2} \left(\sigma_R^2\right)^{1/2} \left(\sigma_C^2\right)^{J/2} \left(\sigma_E^2\right)^{(I+J-1)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I-1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J-1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times\ exp\left[-\frac{SSE}{2\,\sigma_E^2} - \frac{SSR}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC}{2\left(\sigma_E^2 + I\sigma_C^2\right)} - \frac{IJ\left(\bar{y}_{..} - \psi\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right],$$

using the sums of squares presented in Chapter 3.

APPENDIX G

THE NORMALIZING CONSTANT

Let the normalizing constant be defined, by using its inverse,

$$C_1^{-1} = \int_{\Sigma} \int_{-\infty}^{+\infty} g\left(\psi, \sigma \mid \{y_{ij}\}\right) d\psi \, d\sigma$$

where $g\left(\psi, \sigma \mid \{y_{ij}\}\right)$

$$= \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1} exp\left[- \frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} \right]$$

$$\times exp\left[- \frac{IJ\left(\bar{y}_{..} - \psi\right)^2 + \tau\left(\psi - \mu\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right].$$

The integration over ψ is performed analytically. To facilitate the integration, let Q_1 be defined

$$Q_1 = \int_{-\infty}^{+\infty} g\left(\psi, \sigma \mid \{y_{ij}\}\right) d\psi ,$$

so that

$$C_1^{-1} = \int_{\Sigma} Q_1 \, d\sigma .$$

Regarding those terms in the exponent of $g\left(\psi, \sigma \mid \{y_{ij}\}\right)$ which are functions of ψ, completing the squares and arranging terms gives:

$$-\frac{IJ\left(\bar{y}_{..}-\psi\right)^2+\tau\left(\psi-\mu\right)^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}$$

$$=-\frac{IJ\left(\bar{y}_{..}^{\,2}-2\bar{y}_{..}\psi+\psi^2\right)+\tau\left(\psi^2-2\mu\psi+\mu^2\right)}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}$$

$$=-\frac{IJ\bar{y}_{..}^{\,2}}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}+\frac{IJ\bar{y}_{..}\psi}{\sigma_E^2+J\sigma_R^2+I\sigma_C^2}-\frac{IJ\psi^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}$$

$$-\frac{\tau\psi^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}+\frac{\tau\mu\psi}{\sigma_E^2+J\sigma_R^2+I\sigma_C^2}-\frac{\tau\mu^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}$$

$$=-\frac{IJ\bar{y}_{..}^{\,2}+\tau\mu^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}+\left(\frac{IJ\bar{y}_{..}+\tau\mu}{\sigma_E^2+J\sigma_R^2+I\sigma_C^2}\right)\psi-\left(\frac{IJ+\tau}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right)\psi^2 .$$

Removing from the integral terms not involving ψ gives:

$$Q_1 = \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1} exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} \right]$$

$$\times exp\left[-\frac{IJ\,\bar{y}_{..}^{\,2} + \tau\mu^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right] \int_{-\infty}^{+\infty} Q_2 \, d\psi \, ,$$

where

$$Q_2 = exp\left[-\left(\frac{IJ + \tau}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right)\psi^2 + \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} \right)\psi \right].$$

The integral is evaluated using Result (ii) of Appendix 4A:

$$\int_{-\infty}^{+\infty} exp\left[-\alpha X^2 - \beta X \right] dX = \pi^{1/2} \alpha^{-1/2} exp\left[\beta^2 \left(4\alpha \right)^{-1} \right] .$$

Applying this result to the problem at hand, where $X = \psi$,

$$\alpha = \left(\frac{IJ + \tau}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right), \text{ and } \beta = -\left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} \right) \text{gives:}$$

$$\int_{-\infty}^{+\infty} Q_2\, d\psi = \pi^{1/2}\left(\frac{IJ+\tau}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right)^{-1/2} exp\left[\frac{\left[-\left(\frac{IJ\,\bar{y}_{..}+\tau\mu}{\sigma_E^2+J\sigma_R^2+I\sigma_C^2}\right)\right]^2}{\left(\frac{4\left(IJ+\tau\right)}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right)}\right]$$

$$= \left(\frac{2\pi}{IJ+\tau}\right)^{1/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{1/2} exp\left[\frac{\left(IJ\,\bar{y}_{..}+\tau\mu\right)^2}{2\left(IJ+\tau\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right].$$

Substituting the evaluated integral into the expression for Q_1 gives:

$$Q_1 = \left(\sigma_E^2\right)^{-(IJ-I-J+2\alpha_E+3)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I+2\alpha_R+1)/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J+2\alpha_C+1)/2}$$

$$\times\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1} exp\left[-\frac{SSE+2\beta_E^{-1}}{2\sigma_E^2}-\frac{SSR+2\beta_R^{-1}}{2\left(\sigma_E^2+J\sigma_R^2\right)}-\frac{SSC+2\beta_C^{-1}}{2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{IJ\,\bar{y}_{..}^{\,2}+\tau\mu^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$\times\left(\frac{2\pi}{IJ+\tau}\right)^{1/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{1/2} exp\left[\frac{\left(IJ\,\bar{y}_{..}+\tau\mu\right)^2}{2\left(IJ+\tau\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]$$

$$= \left(\frac{2\pi}{IJ + \tau}\right)^{1/2} \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2}$$

$$\times \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2} \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{IJ\,\bar{y}_{..}^{\,2} + \tau\mu^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} + \frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right].$$

Regarding those terms in the exponent that are functions of μ or $\bar{y}_{..}$:

$$-\frac{IJ\,\bar{y}_{..}^{\,2} + \tau\mu^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} + \frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}$$

$$= \frac{-\left(IJ + \tau\right)\left(IJ\,\bar{y}_{..}^{\,2} + \tau\mu^2\right) + \left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}$$

$$= \frac{-\left(IJ\,\bar{y}_{..}\right)^2 - IJ\tau\mu^2 - IJ\tau\,\bar{y}_{..}^{\,2} - \left(\tau\mu\right)^2 + \left(IJ\,\bar{y}_{..}\right)^2 + 2\,IJ\,\bar{y}_{..}\,\tau\mu + \left(\tau\mu\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}$$

$$= \frac{-IJ\tau\mu^2 - \bar{y}_{..}^{\,2} + 2IJ\,\bar{y}_{..}\tau\mu}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}$$

$$= \frac{-IJ\tau\left(\mu^2 + \bar{y}_{..}^{\,2} - 2\bar{y}_{..}\mu\right)}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}$$

$$= \frac{-IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}.$$

Substituting into the expression for Q_1 gives:

$$Q_1 = \left(\frac{2\pi}{IJ + \tau}\right)^{1/2} \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2}$$

$$\times \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2} \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{-IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right].$$

The normalizing constant is given by

$$C_1^{-1} = \left(\frac{2\pi}{IJ + \tau}\right)^{1/2} \int_{\Sigma} g\left(\sigma \mid \{y_{ij}\}\right) d\sigma$$

where $g\left(\sigma \mid \{y_{ij}\}\right)$

$$= \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{-IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right].$$

APPENDIX H

INTEGRATION OVER ψ IN THE JOINT POSTERIOR DISTRIBUTION

The subscript (J+1) is omitted from the variable X in this appendix.

Let

$$Q = \int_{-\infty}^{+\infty} g(\psi)\, d\psi$$

where

$$g(\psi) = C_1 \left(2\pi\sigma_C^2\right)^{-1/2} \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2}$$

$$\times \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2} \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{\left(X - \psi\right)^2}{2\sigma_C^2} - \frac{IJ\left(\bar{y}_{..} - \psi\right)^2 + \tau\left(\psi - \mu\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right].$$

Regarding those terms in the exponent which are functions of ψ, completing the squares and arranging terms gives:

$$-\frac{\left(X - \psi\right)^2}{2\sigma_C^2} - \frac{IJ\left(\bar{y}_{..} - \psi\right)^2 + \tau\left(\psi - \mu\right)^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}$$

$$= -\frac{X^2 - 2X\psi + \psi^2}{2\sigma_C^2} - \frac{IJ\left(\bar{y}_{..}^{\,2} - 2\bar{y}_{..}\psi + \psi^2\right) + \tau\left(\psi^2 - 2\mu\psi + \mu^2\right)}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}$$

$$= -\frac{X^2}{2\sigma_C^2} + \frac{X\psi}{\sigma_C^2} - \frac{\psi^2}{2\sigma_C^2}$$

$$- \frac{IJ\bar{y}_{..}^{\,2}}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} + \frac{IJ\bar{y}_{..}\psi}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} - \frac{IJ\psi^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}$$

$$- \frac{\tau\psi^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} + \frac{\tau\mu\psi}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} - \frac{\tau\mu^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}$$

$$= -\frac{X^2}{2\sigma_C^2} - \frac{IJ\bar{y}_{..}^{\,2} + \tau\mu^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} + \left(\frac{X}{\sigma_C^2} + \frac{IJ\bar{y}_{..} + \tau\mu}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}\right)\psi$$

$$- \left(\frac{1}{2\sigma_C^2} + \frac{IJ + \tau}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right)\psi^2$$

$$= -\frac{X^2}{2\sigma_C^2} - \frac{IJ\bar{y}_{..}^{\,2} + \tau\mu^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} + \left(\frac{X\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + \left(IJ\bar{y}_{..} + \tau\mu\right)\sigma_C^2}{\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right)\psi$$

$$- \left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right)\psi^2.$$

Substituting into Q, and removing from the integral terms not involving ψ gives:

$$Q = C_1\left(2\pi\sigma_C^2\right)^{-1/2}\left(\sigma_E^2\right)^{-(IJ-I-J+2\alpha_E+3)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I+2\alpha_R+1)/2}$$

$$\times\left(\sigma_E^2+I\sigma_C^2\right)^{-(J+2\alpha_C+1)/2}\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1}$$

$$\times\,exp\left[-\frac{SSE+2\beta_E^{-1}}{2\sigma_E^2}-\frac{SSR+2\beta_R^{-1}}{2\left(\sigma_E^2+J\sigma_R^2\right)}-\frac{SSC+2\beta_C^{-1}}{2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times\,exp\left[-\frac{X^2}{2\sigma_C^2}-\frac{IJ\,\bar{y}_{..}^{\,2}+\tau\mu^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right]\int_{-\infty}^{+\infty}Q_1\,d\psi\,,$$

where

$$Q_1 = exp\left[-\left(\frac{\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2}{2\sigma_C^2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right)\psi^2\right]$$

$$\times\,exp\left[+\left(\frac{X\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)+\left(IJ\,\bar{y}_{..}+\tau\mu\right)\sigma_C^2}{\sigma_C^2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right)\psi\right].$$

The integral is evaluated using Result (ii) of Appendix 4A:

$$\int_{-\infty}^{+\infty}exp\left[-\alpha X^2-\beta X\right]dX = \pi^{1/2}\alpha^{-1/2}exp\left[\beta^2\left(4\alpha\right)^{-1}\right].$$

Applying this result to the problem at hand, where $X = \psi$,

$$\alpha = \left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{2\,\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right), \text{ and}$$

$$\beta = -\left(\frac{X\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + \left(IJ\,\bar{y}_{..} + \tau\mu\right)\sigma_C^2}{\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right)$$

gives:

$$\int_{-\infty}^{+\infty} Q_1\, d\psi = \pi^{1/2} \left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{2\,\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right)^{-1/2}$$

$$\times\, exp\left[\frac{\left[-\left(\frac{X\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + \left(IJ\,\bar{y}_{..} + \tau\mu\right)\sigma_C^2}{\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right) \right]^2}{\left(\frac{4\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}{2\,\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right)} \right]$$

$$= \left(2\pi\sigma_C^2\right)^{1/2}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{1/2}\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times\, exp\left[+ \frac{\left[X\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + \left(IJ\,\bar{y}_{..} + \tau\mu\right)\sigma_C^2\right]^2}{2\,\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} \right].$$

Substituting the evaluated integral into the expression for Q gives:

$$Q = C_1 \left(2\pi\sigma_C^2\right)^{-1/2} \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2}$$

$$\times \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2} \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} \right]$$

$$\times exp\left[-\frac{X^2}{2\sigma_C^2} - \frac{IJ\,\bar{y}_{..}^{\;2} + \tau\mu^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right]$$

$$\times \left(2\pi\sigma_C^2\right)^{1/2} \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times exp\left[+ \frac{\left[X\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + \left(IJ\,\bar{y}_{..} + \tau\mu\right)\sigma_C^2\right]^2}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} \right]$$

$$= C_1\left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2}\left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}$$

$$\times\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times\, exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{X^2}{2\sigma_C^2} - \frac{IJ\,\bar{y}_{..}^{\,2} + \tau\mu^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[+\frac{\left[X\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + \left(IJ\,\bar{y}_{..} + \tau\mu\right)\sigma_C^2\right]^2}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}\right].$$

Regarding those terms in the exponent that are functions of X, μ or $\bar{y}_{..}$:

$$-\frac{X^2}{2\sigma_C^2} - \frac{IJ\,\bar{y}_{..}^{\,2} + \tau\mu^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} + \frac{\left[X\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + \left(IJ\,\bar{y}_{..} + \tau\mu\right)\sigma_C^2\right]^2}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$= -\frac{X^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$-\frac{\left(IJ\,\bar{y}_{..}^{\,2} + \tau\mu^2\right)\sigma_C^2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$+\frac{\left[X\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + \left(IJ\,\bar{y}_{..} + \tau\mu\right)\sigma_C^2\right]^2}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$= - \frac{X^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + \left(IJ + \tau\right)\sigma_C^2\right]}{2\,\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{\left(IJ\,\bar{y}_{..}^{\,2} + \tau\mu^2\right)\sigma_C^2\left[\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + \left(IJ + \tau\right)\sigma_C^2\right]}{2\,\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$+ \frac{\begin{pmatrix} +\left[X\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\right]^2 \\ +\,2X\left(IJ\,\bar{y}_{..} + \tau\mu\right)\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) \\ +\left[\left(IJ\,\bar{y}_{..} + \tau\mu\right)\sigma_C^2\right]^2 \end{pmatrix}}{2\,\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$= -\frac{\left[X\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\right]^2 + X^2\left(IJ + \tau\right)\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{\left(IJ\,\bar{y}_{..}^{\;2} + \tau\mu^2\right)\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + \left(IJ + \tau\right)\left(IJ\,\bar{y}_{..}^{\;2} + \tau\mu^2\right)\left(\sigma_C^2\right)^2}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$+ \frac{\left[X\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\right]^2}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$+ \frac{2X\left(IJ\,\bar{y}_{..} + \tau\mu\right)\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$+ \frac{\left[\left(IJ\,\bar{y}_{..}\right)^2 + 2IJ\,\bar{y}_{..}\tau\mu + \left(\tau\mu\right)^2\right]\left(\sigma_C^2\right)^2}{2\sigma_C^2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$= -\frac{X^2\left(IJ+\tau\right)}{2\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]}$$

$$-\frac{IJ\,\bar{y}_{..}^{\;2}+\tau\mu^2}{2\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]}$$

$$-\frac{\left(IJ+\tau\right)\left(IJ\,\bar{y}_{..}^{\;2}+\tau\mu^2\right)\sigma_C^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]}$$

$$+\frac{2X\left(IJ\,\bar{y}_{..}+\tau\mu\right)}{2\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]}$$

$$+\frac{\left[\left(IJ\,\bar{y}_{..}\right)^2+2IJ\,\bar{y}_{..}\tau\mu+\left(\tau\mu\right)^2\right]\sigma_C^2}{2\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]}$$

$$= - \frac{X^2\left(IJ + \tau\right) - 2X\left(IJ\,\bar{y}_{..} + \tau\mu\right)}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{IJ\,\bar{y}_{..}^{\;2} + \tau\mu^2}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$+ \frac{\left[\left(IJ\,\bar{y}_{..}\right)^2 + 2IJ\,\bar{y}_{..}\tau\mu + \left(\tau\mu\right)^2 - \left(IJ\,\bar{y}_{..}\right)^2 - IJ\tau\mu^2 - IJ\tau\bar{y}_{..}^{\;2} - \left(\tau\mu\right)^2\right]\sigma_C^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$= - \frac{X^2\left(IJ + \tau\right) - 2X\left(IJ\,\bar{y}_{..} + \tau\mu\right)}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{IJ\,\bar{y}_{..}^{\;2} + \tau\mu^2}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{IJ\tau\left(\mu^2 - 2\,\bar{y}_{..}\mu + \bar{y}_{..}^{\;2}\right)\sigma_C^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$= -\frac{X^2\left(IJ + \tau\right) - 2X\left(IJ\,\bar{y}_{..} + \tau\mu\right)}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{IJ\,\bar{y}_{..}^{\,2} + \tau\mu^2}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{IJ\tau\left(\mu - \bar{y}_{..}\right)^2\sigma_C^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$= - \frac{X^2\left(IJ + \tau\right) - 2X\left(IJ\,\bar{y}_{..} + \tau\mu\right)}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$+ \frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} - \frac{IJ\,\bar{y}_{..}^{\,2} + \tau\mu^2}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{IJ\tau\left(\mu - \bar{y}_{..}\right)^2\sigma_C^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$= - \frac{X^2 - 2X\left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}\right) + \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)^2}{2\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau}\right)}$$

$$+ \frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2 - \left(IJ\,\bar{y}_{..}^{\;2} + \tau\mu^2\right)\left(IJ + \tau\right)}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{IJ\,\tau\left(\mu - \bar{y}_{-}\right)^2\sigma_C^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$= - \frac{\left(X - \frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)^2}{2\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau}\right)}$$

$$+ \frac{\left(IJ\,\bar{y}_{..}\right)^2 + 2\,IJ\,\tau\,\bar{y}_{..}\mu + \left(\tau\mu\right)^2 - \left(IJ\,\bar{y}_{..}\right)^2 - IJ\,\tau\,\bar{y}_{..}^{\;2} - IJ\,\tau\,\mu^2 - \left(\tau\mu\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{IJ\,\tau\left(\mu - \bar{y}_{..}\right)^2\sigma_C^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$= - \frac{\left(X - \frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right)^2}{2\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right)}$$

$$- \frac{IJ\,\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$- \frac{IJ\,\tau\left(\mu - \bar{y}_{..}\right)^2 \sigma_C^2}{2\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}$$

$$= - \frac{\left(X - \frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right)^2}{2\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right)}$$

$$- \frac{IJ\,\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} \left(\frac{1}{IJ + \tau} + \frac{\sigma_C^2}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} \right)$$

$$= -\frac{\left(X - \dfrac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)^2}{2\left(\dfrac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau}\right)}$$

$$- \frac{IJ\,\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right)$$

$$= -\frac{\left(X - \dfrac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)^2}{2\left(\dfrac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau}\right)} - \frac{IJ\,\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}$$

Substituting into the expression for Q gives:

$$Q = C_1\left(\sigma_E^2\right)^{-(IJ-I-J+2\alpha_E+3)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I+2\alpha_R+1)/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J+2\alpha_C+1)/2}$$

$$\times\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times\, exp\left[-\frac{SSE+2\beta_E^{-1}}{2\sigma_E^2}-\frac{SSR+2\beta_R^{-1}}{2\left(\sigma_E^2+J\sigma_R^2\right)}-\frac{SSC+2\beta_C^{-1}}{2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{IJ\tau\left(\mu-\bar{y}_{..}\right)^2}{2\left(IJ+\tau\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}-\frac{\left(X-\dfrac{IJ\bar{y}_{..}+\tau\mu}{IJ+\tau}\right)^2}{2\left(\dfrac{\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2}{IJ+\tau}\right)}\right].$$

APPENDIX I

INTEGRATION OVER X IN THE POSTERIOR EXPECTED VALUE

The subscript (J+1) is omitted from the variable X in this appendix.

Let

$$Q = \int_{\Sigma} \int_{-\infty}^{+\infty} g\left(X, \sigma \,\middle|\, \{y_{ij}\}\right) dX \, d\sigma$$

where $g\left(X, \sigma \,\middle|\, \{y_{ij}\}\right)$

$$= X \cdot C_1 \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times \, exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times \, exp\left[-\frac{IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} - \frac{\left(\dfrac{IJ + \tau}{2}\right)\left(X - \dfrac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)^2}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right]$$

Regarding those terms in the exponent which are functions of X:

$$-\frac{\left(\frac{IJ+\tau}{2}\right)\left(X-\frac{IJ\,\bar{y}_{..}+\tau\mu}{IJ+\tau}\right)^2}{\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2} = -\frac{\left(X-\frac{IJ\,\bar{y}_{..}+\tau\mu}{IJ+\tau}\right)^2}{2\left(\frac{\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2}{IJ+\tau}\right)}$$

$$= -\frac{X^2-2X\left(\frac{IJ\,\bar{y}_{..}+\tau\mu}{IJ+\tau}\right)+\left(\frac{IJ\,\bar{y}_{..}+\tau\mu}{IJ+\tau}\right)^2}{2\left(\frac{\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2}{IJ+\tau}\right)}$$

$$= -\frac{X^2\left(IJ+\tau\right)}{2\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]}+\frac{X\left(IJ\,\bar{y}_{..}+\tau\mu\right)}{\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2}$$

$$-\frac{\left(IJ\,\bar{y}_{..}+\tau\mu\right)^2}{2\left(IJ+\tau\right)\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]}.$$

Substituting and arranging terms gives

$$g\left(X, \sigma \mid \{y_{ij}\}\right)$$

$$= C_1 \left(\sigma_E^2\right)^{-(IJ-I-J+2\alpha_E+3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I+2\alpha_R+1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J+2\alpha_C+1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}\right]$$

$$\times exp\left[-\frac{IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times X \cdot exp\left[-\frac{X^2\left(IJ + \tau\right)}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} - \frac{-X\left(IJ\,\bar{y}_{..} + \tau\mu\right)}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right],$$

The integration over X is performed analytically; let:

$$Q = \int_{\Sigma} \left[Q_1 \int_{-\infty}^{+\infty} Q_2\, dX\right] d\sigma$$

where

$$Q_1 = C_1 \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times exp\left[- \frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} \right]$$

$$\times exp\left[- \frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} \right]$$

$$\times exp\left[- \frac{IJ\,\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right],$$

and

$$Q_2 = X \cdot exp\left[- \frac{X^2\left(IJ + \tau\right)}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} - \frac{-X\left(IJ\,\bar{y}_{..} + \tau\mu\right)}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2} \right].$$

The integration of Q_2 over X is evaluated using Result (iii) of Appendix 4A:

$$\int_{-\infty}^{+\infty} X \cdot exp\left[-\alpha X^2 - \beta X\right] dX = -2^{-1}\,\pi^{1/2}\,\alpha^{-3/2}\,\beta\; exp\left[\beta^2\left(4\alpha\right)^{-1}\right].$$

Applying this result to the problem at hand, where

$$\alpha = \left(\frac{IJ + \tau}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} \right) \text{and } \beta = \left(\frac{-\left(IJ\,\bar{y}_{..} + \tau\mu\right)}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2} \right)$$

gives $\int_{-\infty}^{+\infty} Q_2 \, dX$

$$= -2^{-1}\,\pi^{1/2} \left(\frac{IJ + \tau}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} \right)^{-3/2} \left(\frac{-\left(IJ\,\bar{y}_{..} + \tau\mu\right)}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2} \right)$$

$$\times \exp\left[\left(\frac{-\left(IJ\,\bar{y}_{..} + \tau\mu\right)}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2} \right)^2 \left(\frac{4\left(IJ + \tau\right)}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} \right)^{-1} \right]$$

$$= \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right) \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{1/2}$$

$$\times \exp\left[\frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} \right].$$

Substituting the evaluated integral gives

$$Q_1 \int_{-\infty}^{+\infty} Q_2 \, dX$$

$$= C_1 \left(\sigma_E^2\right)^{-(IJ-I-J+2\alpha_E+3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I+2\alpha_R+1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J+2\alpha_C+1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}\right]$$

$$\times exp\left[-\frac{IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)\left(\frac{2\pi}{IJ + \tau}\right)^{1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{1/2}$$

$$\times exp\left[\frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}\right]$$

$$= \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right) C_1 \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2}$$

$$\times \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2} \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times exp\left[- \frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} \right]$$

$$\times exp\left[- \frac{IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right]$$

$$= \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right) C_1 \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} g\left(\sigma \mid \{y_{ij}\}\right)$$

where $g\left(\sigma \mid \{y_{ij}\}\right)$ is given in Equation (4.12). Substituting into the expression for Q gives:

$$Q = \int_{\Sigma} \left[Q_1 \int_{-\infty}^{+\infty} Q_2 \, dX \right] d\sigma$$

$$= \int_{\Sigma} \left[\left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right) C_1 \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} g\left(\sigma \mid \{y_{ij}\}\right) \right] d\sigma$$

$$= \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right) C_1 \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} \int_{\Sigma} g\left(\sigma \mid \{y_{ij}\}\right) d\sigma$$

$$= \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right) C_1 \cdot C_1^{-1}$$

$$= \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right).$$

APPENDIX J

INTEGRATION OVER X IN THE POSTERIOR VARIANCE

The subscript (J+1) is omitted from the variable X in this appendix.

Let

$$Q = \int_{\Sigma} \int_{-\infty}^{+\infty} g\left(X, \sigma \mid \{y_{ij}\}\right) dX\, d\sigma$$

where $g\left(X, \sigma \mid \{y_{ij}\}\right)$

$$= X^2 \cdot C_1 \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times\, exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{IJ\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} - \frac{\left(\dfrac{IJ + \tau}{2}\right)\left(X - \dfrac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)^2}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right]$$

$$= C_1 \left(\sigma_E^2\right)^{-(IJ-I-J+2\alpha_E+3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I+2\alpha_R+1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J+2\alpha_C+1)/2}$$

$$\times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times exp\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)}\right]$$

$$\times exp\left[-\frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}\right]$$

$$\times exp\left[-\frac{IJ\,\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)}\right]$$

$$\times X^2 \cdot exp\left[-\frac{X^2\left(IJ + \tau\right)}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} - \frac{-X\left(IJ\,\bar{y}_{..} + \tau\mu\right)}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right],$$

after completing the square of the exponent term involving X as in Appendix 4F. The integration over X is performed analytically; let:

$$Q = \int_{\Sigma} \left[Q_1 \int_{-\infty}^{+\infty} Q_2 \, dX\right] d\sigma$$

where

$$Q_1 = C_1\left(\sigma_E^2\right)^{-(IJ-I-J+2\alpha_E+3)/2}\left(\sigma_E^2+J\sigma_R^2\right)^{-(I+2\alpha_R+1)/2}\left(\sigma_E^2+I\sigma_C^2\right)^{-(J+2\alpha_C+1)/2}$$

$$\times\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)^{-1/2}\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^{-1/2}$$

$$\times\, exp\left[-\frac{SSE+2\beta_E^{-1}}{2\sigma_E^2}-\frac{SSR+2\beta_R^{-1}}{2\left(\sigma_E^2+J\sigma_R^2\right)}-\frac{SSC+2\beta_C^{-1}}{2\left(\sigma_E^2+I\sigma_C^2\right)}\right]$$

$$\times\, exp\left[-\frac{\left(IJ\,\bar{y}_{..}+\tau\mu\right)^2}{2\left(IJ+\tau\right)\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]}\right]$$

$$\times\, exp\left[-\frac{IJ\tau\left(\mu-\bar{y}_{..}\right)^2}{2\left(IJ+\tau\right)\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)}\right],$$

and

$$Q_2 = X^2\cdot exp\left[-\frac{X^2\left(IJ+\tau\right)}{2\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]}-\frac{-X\left(IJ\,\bar{y}_{..}+\tau\mu\right)}{\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2}\right].$$

The integration of Q_2 over X is evaluated using Result (iv) of Appendix 4A:

$$\int_{-\infty}^{+\infty} X^2\cdot exp\left[-\alpha X^2-\beta X\right]dX = 4^{-1}\,\pi^{1/2}\,\alpha^{-5/2}\left(\beta^2+2\alpha\right)exp\left[\beta^2\left(4\alpha\right)^{-1}\right].$$

Applying this result to the problem at hand, where

$$\alpha = \left(\frac{IJ + \tau}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau \right)\sigma_C^2 \right]} \right) \text{and } \beta = \left(\frac{-\left(IJ\,\bar{y}_{..} + \tau\mu \right)}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau \right)\sigma_C^2} \right)$$

gives $\int_{-\infty}^{+\infty} Q_2 \, dX$

$$= 4^{-1}\pi^{1/2} \left(\frac{IJ + \tau}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau \right)\sigma_C^2 \right]} \right)^{-5/2}$$

$$\times \left[\left(\frac{-\left(IJ\,\bar{y}_{..} + \tau\mu \right)}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau \right)\sigma_C^2} \right)^2 + \left(\frac{2\left(IJ + \tau \right)}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau \right)\sigma_C^2 \right]} \right) \right]$$

$$\times\, exp \left[\left(\frac{-\left(IJ\,\bar{y}_{..} + \tau\mu \right)}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau \right)\sigma_C^2} \right)^2 \left(\frac{4\left(IJ + \tau \right)}{2\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau \right)\sigma_C^2 \right]} \right)^{-1} \right]$$

$$= \left(2\pi\right)^{1/2}\left(IJ+\tau\right)^{-5/2}\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{5/2}$$

$$\times \left(\frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2 + \left(IJ+\tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \right)$$

$$\times exp\left[\left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right)^2 \left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{2\left(IJ+\tau\right)}\right)\right]$$

$$= \left(2\pi\right)^{1/2}\left(IJ+\tau\right)^{-5/2}\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{5/2}$$

$$\times \left\{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2 + \left(IJ+\tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]\right\}$$

$$\times exp\left[\frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ+\tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]}\right]$$

$$= \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau \right) \sigma_C^2 \right]^{1/2}$$

$$\times \left[\left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right)^2 + \frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau \right) \sigma_C^2}{IJ + \tau} \right]$$

$$\times exp\left[\frac{\left(IJ\,\bar{y}_{..} + \tau\mu \right)^2}{2\left(IJ + \tau \right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau \right) \sigma_C^2 \right]} \right].$$

Substituting the evaluated integral gives

$$Q_1 \int_{-\infty}^{+\infty} Q_2 \, dX$$

$$
\begin{aligned}
= \; & C_1 \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2} \\
& \times \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-1/2} \\
& \times exp\left[- \frac{SSE + 2\,\beta_E^{-1}}{2\,\sigma_E^2} - \frac{SSR + 2\,\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\,\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} \right] \\
& \times exp\left[- \frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} \right] \\
& \times exp\left[- \frac{IJ\,\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right] \\
& \times \left(\frac{2\pi}{IJ + \tau}\right)^{1/2} \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{1/2} \\
& \times \left[\left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau}\right)^2 + \frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right] \\
& \times exp\left[\frac{\left(IJ\,\bar{y}_{..} + \tau\mu\right)^2}{2\left(IJ + \tau\right)\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]} \right]
\end{aligned}
$$

$$= \left[\left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right)^2 + \frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right] C_1 \left(\frac{2\pi}{IJ + \tau} \right)^{1/2}$$

$$\times \left(\sigma_E^2\right)^{-(IJ - I - J + 2\alpha_E + 3)/2} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(I + 2\alpha_R + 1)/2}$$

$$\times \left(\sigma_E^2 + I\sigma_C^2\right)^{-(J + 2\alpha_C + 1)/2} \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-1/2}$$

$$\times \mathit{exp}\left[-\frac{SSE + 2\beta_E^{-1}}{2\sigma_E^2} - \frac{SSR + 2\beta_R^{-1}}{2\left(\sigma_E^2 + J\sigma_R^2\right)} - \frac{SSC + 2\beta_C^{-1}}{2\left(\sigma_E^2 + I\sigma_C^2\right)} \right]$$

$$\times \mathit{exp}\left[-\frac{IJ\,\tau\left(\mu - \bar{y}_{..}\right)^2}{2\left(IJ + \tau\right)\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)} \right]$$

$$= \left[\left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right)^2 + \frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right] C_1 \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} g\left(\sigma \mid \{y_{ij}\}\right),$$

where $g\left(\sigma \mid \{y_{ij}\}\right)$ is given in Equation (4.12). Substituting into the expression for Q gives:

$$Q = \int_{\Sigma} \left[Q_1 \int_{-\infty}^{+\infty} Q_2 \, dX \right] d\sigma$$

$$= \int_{\Sigma} \left[\left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right)^2 C_1 \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} g\left(\sigma \,\middle|\, \{y_{ij}\}\right) \right] d\sigma$$

$$+ \int_{\Sigma} \left[\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right) C_1 \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} g\left(\sigma \,\middle|\, \{y_{ij}\}\right) \right] d\sigma$$

$$= \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right)^2 C_1 \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} \int_{\Sigma} g\left(\sigma \,\middle|\, \{y_{ij}\}\right) d\sigma$$

$$+ C_1 \left(\frac{2\pi}{IJ + \tau} \right)^{1/2} \int_{\Sigma} \left[\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right) g\left(\sigma \,\middle|\, \{y_{ij}\}\right) \right] d\sigma$$

$$= \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right)^2 C_1 \cdot C_1^{-1}$$

$$+ \frac{\left(\frac{2\pi}{IJ + \tau} \right)^{1/2} \int_{\Sigma} \left[\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right) g\left(\sigma \,\middle|\, \{y_{ij}\}\right) \right] d\sigma}{\left(\frac{2\pi}{IJ + \tau} \right)^{1/2} \int_{\Sigma} g\left(\sigma \,\middle|\, \{y_{ij}\}\right) d\sigma}$$

$$= \left(\frac{IJ\,\bar{y}_{..} + \tau\mu}{IJ + \tau} \right)^2 + \frac{\int_{\Sigma} \left[\left(\frac{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}{IJ + \tau} \right) g\left(\sigma \,\middle|\, \{y_{ij}\}\right) \right] d\sigma}{\int_{\Sigma} g\left(\sigma \,\middle|\, \{y_{ij}\}\right) d\sigma} .$$

APPENDIX K

ANALYSIS FOR TYPE 1 DATA SET

When the modes of all three variances have positive values, the approximate value of the integrations in Expression (5.6) is found by applying Equation (5.9) with respect to all three variances.

$$\int_0^\infty \int_0^\infty \int_0^\infty g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right) d\sigma_R^2\, d\sigma_C^2\, d\sigma_E^2$$

$$\approx \left. \frac{exp\left\{-log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]\right\}}{det(\mathbf{H})^{1/2}} \right|_{\sigma_R^2=\sigma_R^{2*},\, \sigma_C^2=\sigma_C^{2*},\, \sigma_E^2=\sigma_E^{2*}},$$

where $\sigma_{.}^{2*}$ denotes the mode, and **H** denotes the (3×3) matrix with (i, j) elements

$$H_{ij} = \left[\frac{\partial^2 log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\partial\left(\sigma_i^2\right)\partial\left(\sigma_j^2\right)}\right], \qquad i, j \in \{R, C, E\}.$$

From Expression (5.6), let

$$g = g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)$$

$$= \left(\sigma_E^2\right)^{-W1}\left(\sigma_E^2 + J\sigma_R^2\right)^{-W2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-W3}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-W4}$$

$$\times\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-W5} exp\left[-\frac{W6}{\sigma_E^2} - \frac{W7}{\sigma_E^2 + J\sigma_R^2}\right]$$

$$\times exp\left[-\frac{W8}{\sigma_E^2 + I\sigma_C^2} - \frac{W9}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} - \frac{W10}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right].$$

Taking the inverse,

$$g^{-1} = \left(\sigma_E^2\right)^{W1}\left(\sigma_E^2 + J\sigma_R^2\right)^{W2}\left(\sigma_E^2 + I\sigma_C^2\right)^{W3}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{W4}$$

$$\times\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{W5} exp\left[+\frac{W6}{\sigma_E^2} + \frac{W7}{\sigma_E^2 + J\sigma_R^2} + \frac{W8}{\sigma_E^2 + I\sigma_C^2}\right]$$

$$\times exp\left[+\frac{W9}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} + \frac{W10}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right].$$

Taking the logarithm of the inverse,

$$log\left(g^{-1}\right) = W1 \cdot log\left(\sigma_E^2\right) + W2 \cdot log\left(\sigma_E^2 + J\sigma_R^2\right) + W3 \cdot log\left(\sigma_E^2 + I\sigma_C^2\right)$$

$$+ W4 \cdot log\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right) + W5 \cdot log\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]$$

$$+ \frac{W6}{\sigma_E^2} + \frac{W7}{\sigma_E^2 + J\sigma_R^2} + \frac{W8}{\sigma_E^2 + I\sigma_C^2} + \frac{W9}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}$$

$$+ \frac{W10}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2} .$$

The first partial derivatives of $log\left(g^{-1}\right)$ are used in the optimization subroutine.

$$\frac{\partial log\left(g^{-1}\right)}{\partial\left(\sigma_R^2\right)}$$

$$= + \frac{J \cdot W2}{\sigma_E^2 + J\sigma_R^2} + \frac{J \cdot W4}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} + \frac{J \cdot W5}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}$$

$$- \frac{J \cdot W7}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} - \frac{J \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{J \cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} .$$

$$\frac{\partial log\left(g^{-1}\right)}{\partial\left(\sigma_C^2\right)}$$

$$= + \frac{I \cdot W3}{\sigma_E^2 + I\sigma_C^2} + \frac{I \cdot W4}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} + \frac{\left(I + IJ + \tau\right) W5}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}$$

$$- \frac{I \cdot W8}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{I \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{\left(I + IJ + \tau\right) W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} .$$

$$\frac{\partial log\left(g^{-1}\right)}{\partial\left(\sigma_E^2\right)}$$

$$= \frac{W1}{\sigma_E^2} + \frac{W2}{\sigma_E^2 + J\sigma_R^2} + \frac{W3}{\sigma_E^2 + I\sigma_C^2} + \frac{W4}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} + \frac{W5}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}$$

$$- \frac{W6}{\left(\sigma_E^2\right)^2} - \frac{W7}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} - \frac{W8}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2}$$

$$- \frac{W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} .$$

The second partial derivatives of $log\left(g^{-1}\right)$ are used in the denominator of the approximation and in the optimization subroutine.

$$\frac{\partial^2 log\left(g^{-1}\right)}{\left[\partial\left(\sigma_R^2\right)\right]^2}$$

$$= -\frac{J^2 \cdot W2}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} - \frac{J^2 \cdot W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{J^2 \cdot W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \frac{2J^2 \cdot W7}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} + \frac{2J^2 \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3} + \frac{2J^2 \cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} .$$

$$\frac{\partial^2 log\left(g^{-1}\right)}{\left[\partial\left(\sigma_C^2\right)\right]^2}$$

$$= -\frac{I^2 \cdot W3}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{I^2 \cdot W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{\left(I + IJ + \tau\right)^2 \cdot W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \frac{2I^2 \cdot W8}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \frac{2I^2 \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3} + \frac{2\left(I + IJ + \tau\right)^2 \cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} .$$

$$\frac{\partial^2 log\left(g^{-1}\right)}{\left[\partial\left(\sigma_E^2\right)\right]^2}$$

$$= -\frac{W1}{\left(\sigma_E^2\right)^2} - \frac{W2}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} - \frac{W3}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2}$$

$$- \frac{W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} + \frac{2 \cdot W6}{\left(\sigma_E^2\right)^3} + \frac{2 \cdot W7}{\left(\sigma_E^2 + J\sigma_R^2\right)^3}$$

$$+ \frac{2 \cdot W8}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \frac{2 \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3} + \frac{2 \cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} .$$

$$\frac{\partial^2 log\left(g^{-1}\right)}{\partial\left(\sigma_R^2\right)\partial\left(\sigma_C^2\right)}$$

$$= -\frac{IJ \cdot W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{J\left(I + IJ + \tau\right) \cdot W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \frac{2IJ \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3} + \frac{2J\left(I + IJ + \tau\right) \cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} .$$

$$\frac{\partial^2 log\left(g^{-1}\right)}{\partial\left(\sigma_R^2\right)\partial\left(\sigma_E^2\right)}$$

$$= -\frac{J\cdot W2}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} - \frac{J\cdot W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{J\cdot W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \frac{2J\cdot W7}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} + \frac{2J\cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3} + \frac{2J\cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} .$$

$$\frac{\partial^2 log\left(g^{-1}\right)}{\partial\left(\sigma_C^2\right)\partial\left(\sigma_E^2\right)}$$

$$= -\frac{I\cdot W3}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{I\cdot W4}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} - \frac{\left(I + IJ + \tau\right)\cdot W5}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \frac{2I\cdot W8}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \frac{2I\cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^3} + \frac{2\left(I + IJ + \tau\right)\cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} .$$

APPENDIX L

ANALYSIS FOR TYPE 2 DATA SET

Section L.1 presents the analysis for approximating the integral in the column variance dimension when the mode is at the zero boundary. Section L.2 presents the derivation of the objective function and its first and second derivatives which are used in the optimization subroutine.

L.1 Apply Equation (5.7) to Expression (5.6) With Respect to σ_C^2

When the mode of the column variance is at the zero boundary, Equation (5.7) is applied to Expression (5.6) with respect to the column variance.

$$\int_0^\infty g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right) d\sigma_C^2 \approx \left. \frac{g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)}{-\dfrac{\partial}{\partial\left(\sigma_C^2\right)} log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)\right]} \right|_{\sigma_C^2=0} .$$

From Expression (5.6), let

$$g = g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)$$

$$= \left(\sigma_E^2\right)^{-W1}\left(\sigma_E^2 + J\sigma_R^2\right)^{-W2}\left(\sigma_E^2 + I\sigma_C^2\right)^{-W3}\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-W4}$$

$$\times\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-W5} exp\left[-\frac{W6}{\sigma_E^2} - \frac{W7}{\sigma_E^2 + J\sigma_R^2}\right].$$

$$\times exp\left[-\frac{W8}{\sigma_E^2 + I\sigma_C^2} - \frac{W9}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} - \frac{W10}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right].$$

Evaluating g at $\sigma_C^2 = 0$,

$$g\big|_{\sigma_C^2 = 0} = \left(\sigma_E^2\right)^{-W1}\left(\sigma_E^2 + J\sigma_R^2\right)^{-W2}\left(\sigma_E^2\right)^{-W3}\left(\sigma_E^2 + J\sigma_R^2\right)^{-W4}\left(\sigma_E^2 + J\sigma_R^2\right)^{-W5}$$

$$\times exp\left[-\frac{W6}{\sigma_E^2} - \frac{W7}{\sigma_E^2 + J\sigma_R^2} - \frac{W8}{\sigma_E^2} - \frac{W9}{\sigma_E^2 + J\sigma_R^2} - \frac{W10}{\sigma_E^2 + J\sigma_R^2}\right]$$

$$= \left(\sigma_E^2\right)^{-(W1 + W3)}\left(\sigma_E^2 + J\sigma_R^2\right)^{-(W2 + W4 + W5)} exp\left[-\frac{W6+W8}{\sigma_E^2} - \frac{W7+W9+W10}{\sigma_E^2 + J\sigma_R^2}\right].$$

The first partial derivative of $log\left(g\right)$ with respect to σ_C^2 is used in the denominator. Taking the logarithm,

$$log(g)$$

$$= -W1 \cdot log(\sigma_E^2) - W2 \cdot log(\sigma_E^2 + J\sigma_R^2) - W3 \cdot log(\sigma_E^2 + I\sigma_C^2)$$

$$- W4 \cdot log(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2) - W5 \cdot log\left[\sigma_E^2 + J\sigma_R^2 + (I + IJ + \tau)\sigma_C^2\right]$$

$$- \frac{W6}{\sigma_E^2} - \frac{W7}{\sigma_E^2 + J\sigma_R^2} - \frac{W8}{\sigma_E^2 + I\sigma_C^2} - \frac{W9}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2}$$

$$- \frac{W10}{\sigma_E^2 + J\sigma_R^2 + (I + IJ + \tau)\sigma_C^2}.$$

Taking the derivative, $\dfrac{\partial log(g)}{\partial(\sigma_C^2)}$

$$= -\frac{I \cdot W3}{\sigma_E^2 + I\sigma_C^2} - \frac{I \cdot W4}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} - \frac{(I + IJ + \tau)W5}{\sigma_E^2 + J\sigma_R^2 + (I + IJ + \tau)\sigma_C^2}$$

$$+ \frac{I \cdot W8}{(\sigma_E^2 + I\sigma_C^2)^2} + \frac{I \cdot W9}{(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2)^2} + \frac{(I + IJ + \tau)W10}{\left[\sigma_E^2 + J\sigma_R^2 + (I + IJ + \tau)\sigma_C^2\right]^2}.$$

Evaluating the derivative at $\sigma_C^2 = 0$,

$$\left.\frac{\partial \log(g)}{\partial\left(\sigma_C^2\right)}\right|_{\sigma_C^2=0}$$

$$= -\frac{I \cdot W3}{\sigma_E^2} - \frac{I \cdot W4}{\sigma_E^2 + J\sigma_R^2} - \frac{(I + IJ + \tau)W5}{\sigma_E^2 + J\sigma_R^2} + \frac{I \cdot W8}{\left(\sigma_E^2\right)^2} + \frac{I \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2\right)^2}$$

$$+ \frac{(I + IJ + \tau)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2}$$

$$= -\frac{I \cdot W3}{\sigma_E^2} - \frac{I \cdot W4 + (I + IJ + \tau)W5}{\sigma_E^2 + J\sigma_R^2} + \frac{I \cdot W8}{\left(\sigma_E^2\right)^2} + \frac{I \cdot W9 + (I + IJ + \tau)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2}.$$

Combining these results for the approximated integral,

$$\left. \frac{g\left(\sigma_C^2, \sigma_R^2, \sigma_E^2\right)}{-\frac{\partial}{\partial\left(\sigma_C^2\right)} log\left[g\left(\sigma_C^2, \sigma_R^2, \sigma_E^2\right)\right]} \right|_{\sigma_C^2 = 0}$$

$$= \left(\sigma_E^2\right)^{-(W1 + W3)} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(W2 + W4 + W5)} exp\left[-\frac{W6 + W8}{\sigma_E^2} - \frac{W7 + W9 + W10}{\sigma_E^2 + J\sigma_R^2}\right]$$

$$\times \begin{pmatrix} \frac{I \cdot W3}{\sigma_E^2} + \frac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ -\frac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \frac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{pmatrix}^{-1}$$

$$= \tilde{g}\left(\sigma_R^2, \sigma_E^2\right), \text{ say.}$$

L.2 Apply Equation (5.9) to $\tilde{g}$ With Respect to σ_R^2 and σ_E^2

When the modes of the row and error variances have positive values, apply Equation (5.9) to the result from Step 1 with respect to the row and error variances.

$$\int_0^\infty \int_0^\infty \tilde{g}\left(\sigma_R^2, \sigma_E^2\right) d\sigma_R^2 \, d\sigma_E^2 \approx \left. \frac{exp\left\{-log\left[\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)^{-1}\right]\right\}}{det(\mathbf{H})^{1/2}} \right|_{\sigma_R^2 = \sigma_R^{2*},\, \sigma_E^2 = \sigma_E^{2*}},$$

where $\sigma_.^{2*}$ denotes the mode, and **H** denotes the (2×2) matrix with (i, j) elements

$$H_{ij} = \left[\frac{\partial^2 log\left[g\left(\sigma_R^2, \sigma_E^2\right)^{-1}\right]}{\partial\left(\sigma_i^2\right)\partial\left(\sigma_j^2\right)} \right] \qquad i, j \in \{R, E\} .$$

Let

$$g = \tilde{g}\left(\sigma_R^2, \sigma_E^2\right)$$

$$= \left(\sigma_E^2\right)^{-(W1 + W3)} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(W2 + W4 + W5)} exp\left[- \frac{W6 + W8}{\sigma_E^2} - \frac{W7 + W9 + W10}{\sigma_E^2 + J\sigma_R^2} \right]$$

$$\times \left(\begin{array}{c} \frac{I \cdot W3}{\sigma_E^2} + \frac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \frac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \frac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right)^{-1} .$$

Taking the inverse, g^{-1}

$$= \left(\sigma_E^2\right)^{W1 + W3} \left(\sigma_E^2 + J\sigma_R^2\right)^{W2 + W4 + W5} exp\left[\frac{W6 + W8}{\sigma_E^2} + \frac{W7 + W9 + W10}{\sigma_E^2 + J\sigma_R^2} \right]$$

$$\times \left(\begin{array}{c} \frac{I \cdot W3}{\sigma_E^2} + \frac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \frac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \frac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right) .$$

Taking the logarithm of the inverse, $log\left(g^{-1}\right)$

$$= \left(W1 + W3\right) log\left(\sigma_E^2\right) + \left(W2 + W4 + W5\right) log\left(\sigma_E^2 + J\sigma_R^2\right) + \frac{W6 + W8}{\sigma_E^2}$$

$$+ \frac{W7 + W9 + W10}{\sigma_E^2 + J\sigma_R^2} + log\left(\begin{array}{c} + \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right) W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \dfrac{I \cdot W\,8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right) W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right).$$

The first partial derivatives of $log\left(g^{-1}\right)$ are used in the optimization subroutine.

$$\frac{\partial \log\left(g^{-1}\right)}{\partial\left(\sigma_R^2\right)}$$

$$= + \frac{J\left(W2 + W4 + W5\right)}{\sigma_E^2 + J\sigma_R^2} - \frac{J\left(W7 + W9 + W10\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^2}$$

$$+ \left(- \frac{J\left[I \cdot W4 + \left(I + IJ + \tau\right)W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2J\left[I \cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} \right)$$

$$\times \left(\begin{array}{l} + \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \dfrac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right)^{-1} .$$

$$\frac{\partial log\left(g^{-1}\right)}{\partial\left(\sigma_E^2\right)}$$

$$= + \frac{W1 + W3}{\sigma_E^2} + \frac{W2 + W4 + W5}{\sigma_E^2 + J\sigma_R^2} - \frac{W6 + W8}{\left(\sigma_E^2\right)^2} - \frac{W7 + W9 + W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2}$$

$$+ \left(\begin{array}{c} - \frac{I \cdot W3}{\left(\sigma_E^2\right)^2} - \frac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \\ + \frac{2I \cdot W8}{\left(\sigma_E^2\right)^3} + \frac{2\left[I \cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} \end{array} \right)$$

$$\times \left(\begin{array}{c} + \frac{I \cdot W3}{\sigma_E^2} + \frac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ - \frac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \frac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right)^{-1} .$$

The second partial derivatives of $log\left(g^{-1}\right)$ are used in the denominator of the approximation and in the optimization subroutine.

$$
\frac{\partial^2 log\left(g^{-1}\right)}{\left[\partial\left(\sigma_R^2\right)\right]^2}
$$

$$
= -\frac{J^2\left(W2 + W4 + W5\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2J^2\left(W7 + W9 + W10\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^3}
$$

$$
+ \left(\frac{2J\left[I \cdot W4 + \left(I + IJ + \tau\right)W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} - \frac{6J\left[I \cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^4} \right)
$$

$$
\times \left(\begin{array}{c} + \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \dfrac{I \cdot W\,8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right)^{-1}
$$

$$
- \left(- \frac{J\left[I \cdot W4 + \left(I + IJ + \tau\right)W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2J\left[I \cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} \right)^2
$$

$$
\times \left(\begin{array}{c} + \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \dfrac{I \cdot W\,8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right)^{-2} .
$$

$$\frac{\partial^2 log\left(g^{-1}\right)}{\partial\left(\sigma_R^2\right)\partial\left(\sigma_E^2\right)}$$

$$= -\frac{J\left(W2 + W4 + W5\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2J\left(W7 + W9 + W10\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^3}$$

$$+\left(\frac{2J\left[I\cdot W4 + \left(I + IJ + \tau\right)W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} - \frac{6J\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^4}\right)$$

$$\times\left(\begin{array}{c} + \frac{I\cdot W3}{\sigma_E^2} + \frac{I\cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2 + J\sigma_R^2} \\ \\ - \frac{I\cdot W8}{\left(\sigma_E^2\right)^2} - \frac{I\cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array}\right)^{-1}$$

$$-\left(-\frac{J\left[I\cdot W4 + \left(I + IJ + \tau\right)W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2J\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3}\right)$$

$$\times\left(\begin{array}{c} -\frac{I\cdot W3}{\left(\sigma_E^2\right)^2} - \frac{I\cdot W4 + \left(I + IJ + \tau\right)W5}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \\ \\ + \frac{2I\cdot W8}{\left(\sigma_E^2\right)^3} + \frac{2\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} \end{array}\right)$$

$$\times \left(+ \frac{I \cdot W3}{\sigma_E^2} + \frac{I \cdot W4 + \left(I + IJ + \tau\right) W5}{\sigma_E^2 + J\sigma_R^2} - \frac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \frac{I \cdot W9 + \left(I + IJ + \tau\right) W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \right)^{-2} .$$

$$\frac{\partial^2 log\left(g^{-1}\right)}{\left[\partial\left(\sigma_E^2\right)\right]^2}$$

$$= - \frac{W1 + W3}{\left(\sigma_E^2\right)^2} - \frac{W2 + W4 + W5}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2\left(W6 + W8\right)}{\left(\sigma_E^2\right)^3} + \frac{2\left(W7 + W9 + W10\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^3}$$

$$+ \left(+ \frac{2I \cdot W3}{\left(\sigma_E^2\right)^3} + \frac{2\left[I \cdot W4 + \left(I + IJ + \tau\right) W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} - \frac{6I \cdot W8}{\left(\sigma_E^2\right)^4} - \frac{6\left[I \cdot W9 + \left(I + IJ + \tau\right) W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^4} \right)$$

$$\times \left(+ \frac{I \cdot W3}{\sigma_E^2} + \frac{I \cdot W4 + \left(I + IJ + \tau\right) W5}{\sigma_E^2 + J\sigma_R^2} - \frac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \frac{I \cdot W9 + \left(I + IJ + \tau\right) W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \right)^{-1}$$

$$
-\left(
\begin{aligned}
&-\frac{I\cdot W3}{\left(\sigma_E^2\right)^2}-\frac{I\cdot W4+\left(I+IJ+\tau\right)W5}{\left(\sigma_E^2+J\sigma_R^2\right)^2}\\
&+\frac{2I\cdot W8}{\left(\sigma_E^2\right)^3}+\frac{2\left[I\cdot W9+\left(I+IJ+\tau\right)W10\right]}{\left(\sigma_E^2+J\sigma_R^2\right)^3}
\end{aligned}
\right)^2
$$

$$
\times\left(
\begin{aligned}
&+\frac{I\cdot W3}{\sigma_E^2}+\frac{I\cdot W4+\left(I+IJ+\tau\right)W5}{\sigma_E^2+J\sigma_R^2}\\
&-\frac{I\cdot W8}{\left(\sigma_E^2\right)^2}-\frac{I\cdot W9+\left(I+IJ+\tau\right)W10}{\left(\sigma_E^2+J\sigma_R^2\right)^2}
\end{aligned}
\right)^{-2}.
$$

APPENDIX M

ANALYSIS FOR TYPE 3 DATA SET

Section M.1 presents the analysis for approximating the integral in the row variance dimension when the mode is at the zero boundary. Section M.2 presents the derivation of the objective function and its first and second derivatives which are used in the optimization subroutine.

M.1 Apply Equation (5.7) to Expression (5.6) With Respect to σ_R^2

$$\int_0^\infty g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right) d\sigma_R^2 \approx \left. \frac{g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)}{-\dfrac{\partial}{\partial\left(\sigma_R^2\right)} log\left[g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)\right]} \right|_{\sigma_r^2 = 0} .$$

From Expression (5.6), let

$$g = g\left(\sigma_R^2, \sigma_C^2, \sigma_E^2\right)$$

$$= \left(\sigma_E^2\right)^{-W1} \left(\sigma_E^2 + J\sigma_R^2\right)^{-W2} \left(\sigma_E^2 + I\sigma_C^2\right)^{-W3} \left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^{-W4}$$

$$\times \left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-W5} exp\left[-\frac{W6}{\sigma_E^2} - \frac{W7}{\sigma_E^2 + J\sigma_R^2} - \frac{W8}{\sigma_E^2 + I\sigma_C^2}\right]$$

$$\times exp\left[-\frac{W9}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} - \frac{W10}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right].$$

Evaluating g at $\sigma_R^2 = 0$,

$g\big|_{\sigma_R^2=0}$

$$= \left(\sigma_E^2\right)^{-W1}\left(\sigma_E^2\right)^{-W2}\left(\sigma_E^2+I\sigma_C^2\right)^{-W3}\left(\sigma_E^2+I\sigma_C^2\right)^{-W4}\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^{-W5}$$

$$\times \exp\left[-\frac{W6}{\sigma_E^2}-\frac{W7}{\sigma_E^2}-\frac{W8}{\sigma_E^2+I\sigma_C^2}-\frac{W9}{\sigma_E^2+I\sigma_C^2}-\frac{W10}{\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2}\right]$$

$$= \left(\sigma_E^2\right)^{-(W1+W2)}\left(\sigma_E^2+I\sigma_C^2\right)^{-(W3+W4)}\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^{-W5}$$

$$\times \exp\left[-\frac{W6+W7}{\sigma_E^2}-\frac{W8+W9}{\sigma_E^2+I\sigma_C^2}-\frac{W10}{\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2}\right].$$

Taking the logarithm, $\log\left(g\right)$

$$= -W1\cdot\log\left(\sigma_E^2\right)-W2\cdot\log\left(\sigma_E^2+J\sigma_R^2\right)-W3\cdot\log\left(\sigma_E^2+I\sigma_C^2\right)$$

$$-W4\cdot\log\left(\sigma_E^2+J\sigma_R^2+I\sigma_C^2\right)-W5\cdot\log\left[\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2\right]$$

$$-\frac{W6}{\sigma_E^2}-\frac{W7}{\sigma_E^2+J\sigma_R^2}-\frac{W8}{\sigma_E^2+I\sigma_C^2}-\frac{W9}{\sigma_E^2+J\sigma_R^2+I\sigma_C^2}$$

$$-\frac{W10}{\sigma_E^2+J\sigma_R^2+\left(I+IJ+\tau\right)\sigma_C^2}.$$

The first partial derivative of $\log\left(g\right)$ with respect to σ_R^2 is used in the denominator;

$$\frac{\partial log\left(g\right)}{\partial\left(\sigma_R^2\right)}$$

$$= -\frac{J \cdot W2}{\sigma_E^2 + J\sigma_R^2} - \frac{J \cdot W4}{\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2} - \frac{J \cdot W5}{\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2}$$

$$+ \frac{J \cdot W7}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{J \cdot W9}{\left(\sigma_E^2 + J\sigma_R^2 + I\sigma_C^2\right)^2} + \frac{J \cdot W10}{\left[\sigma_E^2 + J\sigma_R^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} .$$

Evaluating the derivative at $\sigma_R^2 = 0$,

$$\left.\frac{\partial log\left(g\right)}{\partial\left(\sigma_R^2\right)}\right|_{\sigma_R^2 = 0}$$

$$= -\frac{J \cdot W2}{\sigma_E^2} - \frac{J \cdot W4}{\sigma_E^2 + I\sigma_C^2} - \frac{J \cdot W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} + \frac{J \cdot W7}{\left(\sigma_E^2\right)^2} + \frac{J \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2}$$

$$+ \frac{J \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} .$$

Combining these results for the approximated integral,

$$\left. \frac{g\left(\sigma_C^2, \sigma_R^2, \sigma_E^2\right)}{-\frac{\partial}{\partial\left(\sigma_R^2\right)} log\left[g\left(\sigma_C^2, \sigma_R^2, \sigma_E^2\right)\right]} \right|_{\sigma_R^2=0}$$

$$= \left(\sigma_E^2\right)^{-(W1+W2)} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(W3+W4)} \left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-W5}$$

$$\times exp\left[-\frac{W6+W7}{\sigma_E^2} - \frac{W8+W9}{\sigma_E^2 + I\sigma_C^2} - \frac{W10}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right]$$

$$\times \left(\begin{array}{c} + \frac{J\cdot W2}{\sigma_E^2} + \frac{J\cdot W4}{\sigma_E^2 + I\sigma_C^2} + \frac{J\cdot W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \frac{J\cdot W7}{\left(\sigma_E^2\right)^2} - \frac{J\cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{J\cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right)^{-1}$$

$$= \tilde{g}\left(\sigma_C^2, \sigma_E^2\right), \text{ say.}$$

<u>M.2 Apply Result 5.4 to $\tilde{g}$ With Respect to</u> σ_C^2 <u>and</u> σ_E^2

$$\int_0^\infty \int_0^\infty \tilde{g}\left(\sigma_C^2, \sigma_E^2\right) d\sigma_C^2 \, d\sigma_E^2 \approx \left. \frac{-exp\left\{log\left[\tilde{g}\left(\sigma_C^2, \sigma_E^2\right)^{-1}\right]\right\}}{det(\mathbf{H})^{1/2}} \right|_{\sigma_C^2=\sigma_C^{2*}, \sigma_E^2=\sigma_E^{2*}},$$

where $\sigma_.^{2*}$ denotes the mode, and **H** denotes the (2×2) matrix with (i,j) elements

$$H_{ij} = \left[\frac{\partial^2 log\left[\tilde{g}\left(\sigma_C^2, \sigma_E^2\right)^{-1}\right]}{\partial\left(\sigma_i^2\right)\partial\left(\sigma_j^2\right)} \right], \qquad i, j \in \{C, E\} .$$

Let

$$g = \tilde{g}\left(\sigma_C^2, \sigma_E^2\right)$$

$$= \left(\sigma_E^2\right)^{-(W1 + W2)} \left(\sigma_E^2 + I\sigma_C^2\right)^{-(W3 + W4)} \left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{-W5}$$

$$\times exp\left[- \frac{W6 + W7}{\sigma_E^2} - \frac{W8 + W9}{\sigma_E^2 + I\sigma_C^2} - \frac{W10}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \right]$$

$$\times \left(\begin{array}{c} + \dfrac{J \cdot W2}{\sigma_E^2} + \dfrac{J \cdot W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{J \cdot W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ \\ - \dfrac{J \cdot W7}{\left(\sigma_E^2\right)^2} - \dfrac{J \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{J \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right)^{-1}$$

Taking the inverse,

$$g^{-1} = \left(\sigma_E^2\right)^{(W1 + W2)} \left(\sigma_E^2 + I\sigma_C^2\right)^{(W3 + W4)} \left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^{W5}$$

$$\times\, exp\left[+ \frac{W6 + W7}{\sigma_E^2} + \frac{W8 + W9}{\sigma_E^2 + I\sigma_C^2} + \frac{W10}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2}\right]$$

$$\times \left(\begin{array}{c} + \dfrac{J \cdot W2}{\sigma_E^2} + \dfrac{J \cdot W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{J \cdot W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \dfrac{J \cdot W7}{\left(\sigma_E^2\right)^2} - \dfrac{J \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{J \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right).$$

Taking the logarithm of the inverse, $log\left(g^{-1}\right)$

$$= \left(W1 + W2\right) log\left(\sigma_E^2\right) + \left(W3 + W4\right) log\left(\sigma_E^2 + I\sigma_C^2\right)$$

$$+ W5 \cdot log\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right] + \frac{W6 + W7}{\sigma_E^2} + \frac{W8 + W9}{\sigma_E^2 + I\sigma_C^2}$$

$$+ \frac{W10}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2}$$

$$+ log \left(\begin{array}{c} + \dfrac{J \cdot W2}{\sigma_E^2} + \dfrac{J \cdot W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{J \cdot W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \dfrac{J \cdot W7}{\left(\sigma_E^2\right)^2} - \dfrac{J \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{J \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right).$$

The first partial derivatives of $log\left(g^{-1}\right)$ are used in the optimization subroutine.

$$\frac{\partial log\left(g^{-1}\right)}{\partial\left(\sigma_C^2\right)}$$

$$= +\frac{I\cdot\left(W3 + W4\right)}{\sigma_E^2 + I\sigma_C^2} + \frac{\left(I + IJ + \tau\right)W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} - \frac{I\cdot\left(W8 + W9\right)}{\left(\sigma_E^2 + I\sigma_C^2\right)^2}$$

$$- \frac{\left(I + IJ + \tau\right)W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \left(\begin{array}{l} - \dfrac{IJ\cdot W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{\left(I + IJ + \tau\right)J\cdot W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \\ + \dfrac{2IJ\cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \dfrac{2\left(I + IJ + \tau\right)J\cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} \end{array} \right)$$

$$\times \left(\begin{array}{l} + \dfrac{J\cdot W2}{\sigma_E^2} + \dfrac{J\cdot W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{J\cdot W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \dfrac{J\cdot W7}{\left(\sigma_E^2\right)^2} - \dfrac{J\cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{J\cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right)^{-1}$$

$$= +\frac{I\cdot(W3+W4)}{\sigma_E^2+I\sigma_C^2}+\frac{(I+IJ+\tau)W5}{\sigma_E^2+(I+IJ+\tau)\sigma_C^2}-\frac{I\cdot(W8+W9)}{(\sigma_E^2+I\sigma_C^2)^2}$$

$$-\frac{(I+IJ+\tau)W10}{\left[\sigma_E^2+(I+IJ+\tau)\sigma_C^2\right]^2}$$

$$+\left(\begin{array}{l}-\dfrac{I\cdot W4}{(\sigma_E^2+I\sigma_C^2)^2}-\dfrac{(I+IJ+\tau)W5}{\left[\sigma_E^2+(I+IJ+\tau)\sigma_C^2\right]^2}\\[2ex]+\dfrac{2I\cdot W9}{(\sigma_E^2+I\sigma_C^2)^3}+\dfrac{2(I+IJ+\tau)W10}{\left[\sigma_E^2+(I+IJ+\tau)\sigma_C^2\right]^3}\end{array}\right)$$

$$\times\left(\begin{array}{l}+\dfrac{W2}{\sigma_E^2}+\dfrac{W4}{\sigma_E^2+I\sigma_C^2}+\dfrac{W5}{\sigma_E^2+(I+IJ+\tau)\sigma_C^2}\\[2ex]-\dfrac{W7}{(\sigma_E^2)^2}-\dfrac{W9}{(\sigma_E^2+I\sigma_C^2)^2}-\dfrac{W10}{\left[\sigma_E^2+(I+IJ+\tau)\sigma_C^2\right]^2}\end{array}\right)^{-1}.$$

$$\frac{\partial log\left(g^{-1}\right)}{\partial\left(\sigma_E^2\right)}$$

$$= \frac{W1 + W2}{\sigma_E^2} + \frac{W3 + W4}{\sigma_E^2 + I\sigma_C^2} + \frac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} - \frac{W6 + W7}{\left(\sigma_E^2\right)^2} - \frac{W8 + W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2}$$

$$- \frac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \left(\begin{array}{c} - \frac{J\cdot W2}{\left(\sigma_E^2\right)^2} - \frac{J\cdot W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{J\cdot W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \\ + \frac{2J\cdot W7}{\left(\sigma_E^2\right)^3} + \frac{2J\cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \frac{2J\cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} \end{array} \right)$$

$$\times \left(\begin{array}{c} + \frac{J\cdot W2}{\sigma_E^2} + \frac{J\cdot W4}{\sigma_E^2 + I\sigma_C^2} + \frac{J\cdot W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \frac{J\cdot W7}{\left(\sigma_E^2\right)^2} - \frac{J\cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{J\cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right)^{-1}$$

$$= \frac{W1 + W2}{\sigma_E^2} + \frac{W3 + W4}{\sigma_E^2 + I\sigma_C^2} + \frac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} - \frac{W6 + W7}{\left(\sigma_E^2\right)^2} - \frac{W8 + W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2}$$

$$- \frac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}$$

$$+ \left(\begin{array}{l} - \dfrac{W2}{\left(\sigma_E^2\right)^2} - \dfrac{W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \\ + \dfrac{2 \cdot W7}{\left(\sigma_E^2\right)^3} + \dfrac{2 \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \dfrac{2 \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} \end{array} \right)$$

$$\times \left(\begin{array}{l} + \dfrac{W2}{\sigma_E^2} + \dfrac{W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \dfrac{W7}{\left(\sigma_E^2\right)^2} - \dfrac{W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right)^{-1} .$$

The second partial derivatives of $log\left(g^{-1}\right)$ are used in the denominator of the approximation and in the optimization subroutine.

$$\frac{\partial^2 log\left(g^{-1}\right)}{\left[\partial\left(\sigma_C^2\right)\right]^2}$$

$$= -\frac{I^2\cdot\left(W3 + W4\right)}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{\left(I + IJ + \tau\right)^2\cdot W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} + \frac{2I^2\cdot\left(W8 + W9\right)}{\left(\sigma_E^2 + I\sigma_C^2\right)^3}$$

$$+ \frac{2\left(I + IJ + \tau\right)^2\cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3}$$

$$+ \left(\begin{array}{c} + \dfrac{2I^2\cdot W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \dfrac{2\left(I + IJ + \tau\right)^2\cdot W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} \\ - \dfrac{6I^2\cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^4} - \dfrac{6\left(I + IJ + \tau\right)^2\cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^4} \end{array}\right)$$

$$\times \left(\begin{array}{c} + \dfrac{W2}{\sigma_E^2} + \dfrac{W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \dfrac{W7}{\left(\sigma_E^2\right)^2} - \dfrac{W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array}\right)^{-1}$$

$$
-\left(
\begin{array}{l}
-\dfrac{I\cdot W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{\left(I + IJ + \tau\right)W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \\[2em]
+\dfrac{2I\cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \dfrac{2\left(I + IJ + \tau\right)W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3}
\end{array}
\right)^2
$$

$$
\times\left(
\begin{array}{l}
+\dfrac{W2}{\sigma_E^2} + \dfrac{W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\[2em]
-\dfrac{W7}{\left(\sigma_E^2\right)^2} - \dfrac{W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2}
\end{array}
\right)^{-2} .
$$

$$\frac{\partial^2 log\left(g^{-1}\right)}{\partial\left(\sigma_C^2\right)\partial\left(\sigma_E^2\right)}$$

$$= -\frac{I\cdot\left(W3+W4\right)}{\left(\sigma_E^2+I\sigma_C^2\right)^2} - \frac{\left(I+IJ+\tau\right)W5}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^2} + \frac{2I\cdot\left(W8+W9\right)}{\left(\sigma_E^2+I\sigma_C^2\right)^3}$$

$$+\frac{2\left(I+IJ+\tau\right)W10}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^3}$$

$$+\left(\begin{array}{l} +\dfrac{2I\cdot W4}{\left(\sigma_E^2+I\sigma_C^2\right)^3} + \dfrac{2\left(I+IJ+\tau\right)W5}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^3} \\ -\dfrac{6I\cdot W9}{\left(\sigma_E^2+I\sigma_C^2\right)^4} - \dfrac{6\left(I+IJ+\tau\right)W10}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^4} \end{array}\right)$$

$$\times\left(\begin{array}{l} +\dfrac{W2}{\sigma_E^2} + \dfrac{W4}{\sigma_E^2+I\sigma_C^2} + \dfrac{W5}{\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2} \\ -\dfrac{W7}{\left(\sigma_E^2\right)^2} - \dfrac{W9}{\left(\sigma_E^2+I\sigma_C^2\right)^2} - \dfrac{W10}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^2} \end{array}\right)^{-1}$$

$$
-\left(\begin{array}{l}
-\dfrac{W2}{\left(\sigma_E^2\right)^2}-\dfrac{W4}{\left(\sigma_E^2+I\sigma_C^2\right)^2}-\dfrac{W5}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^2} \\
+\dfrac{2\cdot W7}{\left(\sigma_E^2\right)^3}+\dfrac{2\cdot W9}{\left(\sigma_E^2+I\sigma_C^2\right)^3}+\dfrac{2\cdot W10}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^3}
\end{array}\right)
$$

$$
\times\left(\begin{array}{l}
-\dfrac{I\cdot W4}{\left(\sigma_E^2+I\sigma_C^2\right)^2}-\dfrac{\left(I+IJ+\tau\right)W5}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^2} \\
+\dfrac{2I\cdot W9}{\left(\sigma_E^2+I\sigma_C^2\right)^3}+\dfrac{2\left(I+IJ+\tau\right)W10}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^3}
\end{array}\right)
$$

$$
\times\left(\begin{array}{l}
+\dfrac{W2}{\sigma_E^2}+\dfrac{W4}{\sigma_E^2+I\sigma_C^2}+\dfrac{W5}{\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2} \\
-\dfrac{W7}{\left(\sigma_E^2\right)^2}-\dfrac{W9}{\left(\sigma_E^2+I\sigma_C^2\right)^2}-\dfrac{W10}{\left[\sigma_E^2+\left(I+IJ+\tau\right)\sigma_C^2\right]^2}
\end{array}\right)^{-2}.
$$

$$\frac{\partial^2 log\left(g^{-1}\right)}{\left[\partial\left(\sigma_E^2\right)\right]^2}$$

$$= -\frac{W1 + W2}{\left(\sigma_E^2\right)^2} - \frac{W3 + W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} + \frac{2\left(W6 + W7\right)}{\left(\sigma_E^2\right)^3}$$

$$+ \frac{2\left(W8 + W9\right)}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \frac{2 \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3}$$

$$+ \left(\begin{array}{c} + \frac{2 \cdot W2}{\left(\sigma_E^2\right)^3} + \frac{2 \cdot W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \frac{2 \cdot W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} \\ - \frac{6 \cdot W7}{\left(\sigma_E^2\right)^4} - \frac{6 \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^4} - \frac{6 \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^4} \end{array} \right)$$

$$\times \left(\begin{array}{c} + \frac{W2}{\sigma_E^2} + \frac{W4}{\sigma_E^2 + I\sigma_C^2} + \frac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\ - \frac{W7}{\left(\sigma_E^2\right)^2} - \frac{W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \frac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right)^{-1}$$

$$
- \left(\begin{array}{l} - \dfrac{W2}{\left(\sigma_E^2\right)^2} - \dfrac{W4}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W5}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \\[2ex] + \dfrac{2 \cdot W7}{\left(\sigma_E^2\right)^3} + \dfrac{2 \cdot W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^3} + \dfrac{2 \cdot W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^3} \end{array} \right)^2
$$

$$
\times \left(\begin{array}{l} + \dfrac{W2}{\sigma_E^2} + \dfrac{W4}{\sigma_E^2 + I\sigma_C^2} + \dfrac{W5}{\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2} \\[2ex] - \dfrac{W7}{\left(\sigma_E^2\right)^2} - \dfrac{W9}{\left(\sigma_E^2 + I\sigma_C^2\right)^2} - \dfrac{W10}{\left[\sigma_E^2 + \left(I + IJ + \tau\right)\sigma_C^2\right]^2} \end{array} \right)^{-2} .
$$

APPENDIX N

ANALYSIS FOR TYPE 4 DATA SET

Section N.1 presents the analysis for approximating the integral in the row variance dimension when the mode is at the zero boundary. Section N.2 presents the derivation of the objective function and its first and second derivatives which are used in the optimization subroutine.

N.1 Apply Equation (5.7) to $\tilde{g}$ With Respect to σ_R^2

$$\int_0^\infty \tilde{g}\left(\sigma_R^2, \sigma_E^2\right) d\sigma_R^2 \approx \left. \frac{\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)}{-\frac{\partial}{\partial \sigma_R^2} \log\left[\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)\right]} \right|_{\sigma_R^2 = 0} .$$

Let $g = \tilde{g}\left(\sigma_R^2, \sigma_E^2\right)$; see Appendix L.1 for derivation of this result.

$$g = \left(\sigma_E^2\right)^{-(W1 + W3)} \left(\sigma_E^2 + J\sigma_R^2\right)^{-(W2 + W4 + W5)} exp\left[-\frac{W6 + W8}{\sigma_E^2} - \frac{W7 + W9 + W10}{\sigma_E^2 + J\sigma_R^2}\right]$$

$$\times \left(\frac{\frac{I \cdot W3}{\sigma_E^2} + \frac{I \cdot W4 + \left(I + IJ + \tau\right) W5}{\sigma_E^2 + J\sigma_R^2}}{-\frac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \frac{I \cdot W9 + \left(I + IJ + \tau\right) W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2}} \right)^{-1} .$$

Evaluating g at $\sigma_R^2 = 0$,

$$g \Big|_{\sigma_R^2 = 0}$$

$$= \left(\sigma_E^2\right)^{-(W1 + W3)} \left(\sigma_E^2\right)^{-(W2 + W4 + W5)} exp\left[- \frac{W6 + W8}{\sigma_E^2} - \frac{W7 + W9 + W10}{\sigma_E^2} \right]$$

$$\times \begin{pmatrix} + \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right) W5}{\sigma_E^2} \\ - \dfrac{I \cdot W8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right) W10}{\left(\sigma_E^2\right)^2} \end{pmatrix}^{-1}$$

$$= \left(\sigma_E^2\right)^{-(W1 + W2 + W3 + W4 + W5)} exp\left[- \frac{W6 + W7 + W8 + W9 + W10}{\sigma_E^2} \right]$$

$$\times \left(\frac{I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right) W5}{\sigma_E^2} - \frac{I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right) W10}{\left(\sigma_E^2\right)^2} \right)^{-1}$$

$$= \left(\sigma_E^2\right)^{-(W1 + W2 + W3 + W4 + W5 - 2)} exp\left[- \frac{W6 + W7 + W8 + W9 + W10}{\sigma_E^2} \right]$$

$$\times \begin{pmatrix} \left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right) W5\right]\left(\sigma_E^2\right) \\ - \left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right) W10\right] \end{pmatrix}^{-1} .$$

The first partial derivative of $log\left(g\right)$ with respect to σ_R^2 is used in the denominator of the approximation. Taking the logarithm, $log\left(g\right)$

$$= -\left(W1 + W3\right) log\left(\sigma_E^2\right) - \left(W2 + W4 + W5\right) log\left(\sigma_E^2 + J\ \sigma_R^2\right) - \frac{W6 + W8}{\sigma_E^2}$$

$$- \frac{W7 + W9 + W10}{\sigma_E^2 + J\sigma_R^2} - log\left(\begin{array}{c} + \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right) W5}{\sigma_E^2 + J\sigma_R^2} \\ - \dfrac{I \cdot W\,8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right) W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right).$$

Taking the derivative, $\dfrac{\partial log\left(g\right)}{\partial\left(\sigma_R^2\right)}$

$$= -\frac{J\left(W2 + W4 + W5\right)}{\sigma_E^2 + J\sigma_R^2} + \frac{J\left(W7 + W9 + W10\right)}{\left(\sigma_E^2 + J\sigma_R^2\right)^2}$$

$$- \left(- \frac{J\left[I \cdot W4 + \left(I + IJ + \tau\right) W5\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} + \frac{2J\left[I \cdot W9 + \left(I + IJ + \tau\right) W10\right]}{\left(\sigma_E^2 + J\sigma_R^2\right)^3} \right)$$

$$\times \left(\begin{array}{c} + \dfrac{I \cdot W3}{\sigma_E^2} + \dfrac{I \cdot W4 + \left(I + IJ + \tau\right) W5}{\sigma_E^2 + J\sigma_R^2} \\ - \dfrac{I \cdot W\,8}{\left(\sigma_E^2\right)^2} - \dfrac{I \cdot W9 + \left(I + IJ + \tau\right) W10}{\left(\sigma_E^2 + J\sigma_R^2\right)^2} \end{array} \right)^{-1}.$$

Evaluating the derivative at $\sigma_R^2 = 0$,

$$\left.\frac{\partial log\left(g\right)}{\partial\left(\sigma_R^2\right)}\right|_{\sigma_R^2=0}$$

$$= -\frac{J\left(W2 + W4 + W5\right)}{\sigma_E^2} + \frac{J\left(W7 + W9 + W10\right)}{\left(\sigma_E^2\right)^2}$$

$$-\left(-\frac{J\left[I\cdot W4 + \left(I + IJ + \tau\right)W5\right]}{\left(\sigma_E^2\right)^2} + \frac{2J\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]}{\left(\sigma_E^2\right)^3}\right)$$

$$\times\left(+\frac{I\cdot W3}{\sigma_E^2} + \frac{I\cdot W4 + \left(I + IJ + \tau\right)W5}{\sigma_E^2} - \frac{I\cdot W8}{\left(\sigma_E^2\right)^2} - \frac{I\cdot W9 + \left(I + IJ + \tau\right)W10}{\left(\sigma_E^2\right)^2}\right)^{-1}$$

$$= -J\left(W2 + W4 + W5\right)\left(\sigma_E^2\right)^{-1} + J\left(W7 + W9 + W10\right)\left(\sigma_E^2\right)^{-2}$$

$$+\left(\frac{J\left[I\cdot W4 + \left(I + IJ + \tau\right)W5\right] - 2J\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]\left(\sigma_E^2\right)^{-1}}{\left(\sigma_E^2\right)^2}\right)$$

$$\times\left[\frac{\left(\begin{array}{c} +\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ -\left[I\cdot W8 + I\cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{array}\right)}{\left(\sigma_E^2\right)^2}\right]^{-1}$$

$$= -J\left(W2 + W4 + W5\right)\left(\sigma_E^2\right)^{-1} + J\left(W7 + W9 + W10\right)\left(\sigma_E^2\right)^{-2}$$

$$+\frac{\left(\begin{array}{c} +J\left[I\cdot W4 + \left(I + IJ + \tau\right)W5\right] \\ -2J\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]\left(\sigma_E^2\right)^{-1} \end{array}\right)}{\left(\begin{array}{c} +\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ -\left[I\cdot W8 + I\cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{array}\right)}$$

$$= - \frac{J\left(W2 + W4 + W5\right)\left(\sigma_E^2\right)^{-1}\begin{pmatrix} +\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ -\left[I\cdot W8 + I\cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{pmatrix}}{\begin{pmatrix} +\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ -\left[I\cdot W8 + I\cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{pmatrix}}$$

$$+ \frac{J\left(W7 + W9 + W10\right)\left(\sigma_E^2\right)^{-2}\begin{pmatrix} +\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ -\left[I\cdot W8 + I\cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{pmatrix}}{\begin{pmatrix} +\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ -\left[I\cdot W8 + I\cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{pmatrix}}$$

$$+ \frac{\begin{pmatrix} +J\left[I\cdot W4 + \left(I + IJ + \tau\right)W5\right] \\ -2J\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]\left(\sigma_E^2\right)^{-1} \end{pmatrix}}{\begin{pmatrix} +\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ -\left[I\cdot W8 + I\cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{pmatrix}}$$

$$= - \frac{\left(\begin{array}{c} + J\left(W2 + W4 + W5\right)\left(\sigma_E^2\right)^{-1}\left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ - J\left(W2 + W4 + W5\right)\left(\sigma_E^2\right)^{-1}\left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{array} \right)}{\left(\begin{array}{c} + \left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ - \left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{array} \right)}$$

$$+ \frac{\left(\begin{array}{c} + J\left(W7 + W9 + W10\right)\left(\sigma_E^2\right)^{-2}\left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ - J\left(W7 + W9 + W10\right)\left(\sigma_E^2\right)^{-2}\left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{array} \right)}{\left(\begin{array}{c} + \left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ - \left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{array} \right)}$$

$$+ \frac{\left(\begin{array}{c} + J\left[I \cdot W4 + \left(I + IJ + \tau\right)W5\right] \\ - 2J\left[I \cdot W9 + \left(I + IJ + \tau\right)W10\right]\left(\sigma_E^2\right)^{-1} \end{array} \right)}{\left(\begin{array}{c} + \left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ - \left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{array} \right)}$$

$$
= - \begin{pmatrix}
+J\left(W2+W4+W5\right)\left[I\cdot W3+I\cdot W4+\left(I+IJ+\tau\right)W5\right] \\
-J\left(W2+W4+W5\right)\left[I\cdot W8+I\cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right)^{-1} \\
-J\left(W7+W9+W10\right)\left[I\cdot W3+I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right)^{-1} \\
+J\left(W7+W9+W10\right)\left[I\cdot W8+I\cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right)^{-2} \\
-J\left[I\cdot W4+\left(I+IJ+\tau\right)W5\right] \\
+2J\left[I\cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right)^{-1}
\end{pmatrix}
$$

$$
\times \begin{pmatrix}
+\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\
-\left[I\cdot W8 + I\cdot W9 + \left(I + IJ + \tau\right)W10\right]
\end{pmatrix}^{-1} .
$$

Combining these results for the approximated integral,

$$\left.\frac{\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)}{-\dfrac{\partial}{\partial\left(\sigma_R^2\right)} \log\left[\tilde{g}\left(\sigma_R^2, \sigma_E^2\right)\right]}\right|_{\sigma_R^2 = 0}$$

$$= \left(\sigma_E^2\right)^{-(W1 + W2 + W3 + W4 + W5 - 2)} exp\left[-\frac{W6 + W7 + W8 + W9 + W10}{\sigma_E^2}\right]$$

$$\times \begin{pmatrix} \left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ -\left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} +J\left(W2+W4+W5\right)\left[I \cdot W3+I \cdot W4+\left(I+IJ+\tau\right)W5\right] \\ -J\left(W2+W4+W5\right)\left[I \cdot W8+I \cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right)^{-1} \\ -J\left(W7+W9+W10\right)\left[I \cdot W3+I \cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right)^{-1} \\ +J\left(W7+W9+W10\right)\left[I \cdot W8+I \cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right)^{-2} \\ -J\left[I \cdot W4+\left(I+IJ+\tau\right)W5\right] \\ +2J\left[I \cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right)^{-1} \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} +\left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ -\left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{pmatrix}$$

$$= \left(\sigma_E^2\right)^{-(W1 + W2 + W3 + W4 + W5 - 4)} exp\left[- \frac{W6 + W7 + W8 + W9 + W10}{\sigma_E^2} \right]$$

$$\times \begin{pmatrix} + J(W2+W4+W5)\left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J\left[I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J(W2+W4+W5)\left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ - J(W7+W9+W10)\left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ + 2J\left[I \cdot W9 + \left(I + IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ + J(W7+W9+W10)\left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{pmatrix}^{-1}$$

$$= \tilde{\tilde{g}}\left(\sigma_E^2\right), \text{ say.}$$

<u>N.2 Apply Equation (5.8) to $\tilde{\tilde{g}}$ With Respect to</u> σ_E^2

$$\int_0^\infty \tilde{\tilde{g}}\left(\sigma_E^2\right) d\sigma_E^2 \approx \left. \frac{exp\left\{-log\left[\tilde{\tilde{g}}\left(\sigma_E^2\right)^{-1}\right]\right\}}{\left\{\frac{\partial^2}{\left(\partial\sigma_E^2\right)^2} log\left[\tilde{\tilde{g}}\left(\sigma_E^2\right)^{-1}\right]\right\}^{1/2}} \right|_{\sigma_E^2 = \sigma_E^{2*}},$$

where $\sigma_E^{2\,*}$ denotes the mode of the error variance. Let

$$g = \tilde{\tilde{g}}\left(\sigma_E^2\right)$$

$$= \left(\sigma_E^2\right)^{-(W1 + W2 + W3 + W4 + W5 - 4)} exp\left[-\frac{W6 + W7 + W8 + W9 + W10}{\sigma_E^2}\right]$$

$$\times \begin{pmatrix} + J(W2+W4+W5)\left[I \cdot W3+I \cdot W4+\left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J\left[I \cdot W4+\left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J(W2+W4+W5)\left[I \cdot W8+I \cdot W9+\left(I+IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ - J(W7+W9+W10)\left[I \cdot W3+I \cdot W4+\left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ + 2J\left[I \cdot W9+\left(I+IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ + J(W7+W9+W10)\left[I \cdot W8+I \cdot W9+\left(I+IJ + \tau\right)W10\right] \end{pmatrix}^{-1}.$$

Taking the inverse,

$$g^{-1} = \left(\sigma_E^2\right)^{(W1 + W2 + W3 + W4 + W5 - 4)} exp\left[\frac{W6 + W7 + W8 + W9 + W10}{\sigma_E^2}\right]$$

$$\times \begin{pmatrix} + J(W2+W4+W5)\left[I \cdot W3+I \cdot W4+\left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J\left[I \cdot W4+\left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J(W2+W4+W5)\left[I \cdot W8+I \cdot W9+\left(I+IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ - J(W7+W9+W10)\left[I \cdot W3+I \cdot W4+\left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ + 2J\left[I \cdot W9+\left(I+IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ + J(W7+W9+W10)\left[I \cdot W8+I \cdot W9+\left(I+IJ + \tau\right)W10\right] \end{pmatrix}.$$

Taking the logarithm of the inverse,

$$log\left(g^{-1}\right)$$

$$= \left(W1 + W2 + W3 + W4 + W5 - 4\right) log\left(\sigma_E^2\right) + \frac{W6 + W7 + W8 + W9 + W10}{\sigma_E^2}$$

$$+ log\begin{pmatrix} + J(W2+W4+W5)\left[I\cdot W3 + I\cdot W4 + \left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J\left[I\cdot W4 + \left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - J(W2+W4+W5)\left[I\cdot W8 + I\cdot W9 + \left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right) \\ - J(W7+W9+W10)\left[I\cdot W3 + I\cdot W4 + \left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right) \\ + 2J\left[I\cdot W9 + \left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right) \\ + J(W7+W9+W10)\left[I\cdot W8 + I\cdot W9 + \left(I+IJ+\tau\right)W10\right] \end{pmatrix}.$$

The first derivative of $log\left(g^{-1}\right)$ is used in the optimization subroutine.

$$
\frac{d\log\left(g^{-1}\right)}{d\left(\sigma_E^2\right)}
$$

$$
= + \frac{W1 + W2 + W3 + W4 + W5 - 4}{\sigma_E^2} - \frac{W6 + W7 + W8 + W9 + W10}{\left(\sigma_E^2\right)^2}
$$

$$
+ \begin{pmatrix}
+ 2J\,(W2+W4+W5)\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\
- 2J\left[I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\
- J\,(W2+W4+W5)\left[I\cdot W8 + I\cdot W9 + \left(I + IJ + \tau\right)W10\right] \\
- J\,(W7+W9+W10)\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right] \\
+ 2J\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]
\end{pmatrix}
$$

$$
\times \begin{pmatrix}
+ J(W2+W4+W5)\left[I\cdot W3+I\cdot W4+\left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\
- J\left[I\cdot W4+\left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\
- J(W2+W4+W5)\left[I\cdot W8+I\cdot W9+\left(I+IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\
- J(W7+W9+W10)\left[I\cdot W3+I\cdot W4+\left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\
+ 2J\left[I\cdot W9+\left(I+IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\
+ J(W7+W9+W10)\left[I\cdot W8+I\cdot W9+\left(I+IJ + \tau\right)W10\right]
\end{pmatrix}^{-1}
$$

$$= + \frac{W1 + W2 + W3 + W4 + W5 - 4}{\sigma_E^2} - \frac{W6 + W7 + W8 + W9 + W10}{\left(\sigma_E^2\right)^2}$$

$$+ \begin{pmatrix} + 2(W2+W4+W5)\left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ - 2\left[I \cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ - (W2+W4+W5)\left[I \cdot W8 + I \cdot W9 + \left(I + IJ + \tau\right)W10\right] \\ - (W7+W9+W10)\left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right] \\ + 2\left[I \cdot W9 + \left(I + IJ + \tau\right)W10\right] \end{pmatrix}$$

$$\times \begin{pmatrix} + (W2+W4+W5)\left[I \cdot W3 + I \cdot W4 + \left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - \left[I \cdot W4 + \left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - (W2+W4+W5)\left[I \cdot W8 + I \cdot W9 + \left(I+IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ - (W7+W9+W10)\left[I \cdot W3 + I \cdot W4 + \left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ + 2\left[I \cdot W9 + \left(I+IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ + (W7+W9+W10)\left[I \cdot W8 + I \cdot W9 + \left(I+IJ + \tau\right)W10\right] \end{pmatrix}^{-1} .$$

The second derivative of $log\left(g^{-1}\right)$ is used in the denominator of the approximation and in optimization subroutine.

$$\frac{d^2 log\left(g^{-1}\right)}{\left[d\left(\sigma_E^2\right)\right]^2}$$

$$= -\frac{W1 + W2 + W3 + W4 + W5 - 4}{\left(\sigma_E^2\right)^2} + \frac{2\left(W6 + W7 + W8 + W9 + W10\right)}{\left(\sigma_E^2\right)^3}$$

$$+ \begin{pmatrix} + 2\left(W2 + W4 + W5\right)\left[I \cdot W3 + I \cdot W4 + \left(I + IJ + \tau\right)W5\right] \\ - 2\left[I \cdot W4 + \left(I + IJ + \tau\right)W5\right] \end{pmatrix}$$

$$\times \begin{pmatrix} + (W2+W4+W5)\left[I \cdot W3 + I \cdot W4 + \left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - \left[I \cdot W4 + \left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right)^2 \\ - (W2+W4+W5)\left[I \cdot W8 + I \cdot W9 + \left(I+IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ - (W7+W9+W10)\left[I \cdot W3 + I \cdot W4 + \left(I+IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\ + 2\left[I \cdot W9 + \left(I+IJ + \tau\right)W10\right]\left(\sigma_E^2\right) \\ + (W7+W9+W10)\left[I \cdot W8 + I \cdot W9 + \left(I+IJ + \tau\right)W10\right] \end{pmatrix}^{-1}$$

$$
- \begin{pmatrix}
+ 2\left(W2+W4+W5\right)\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\
- 2\left[I\cdot W4 + \left(I + IJ + \tau\right)W5\right]\left(\sigma_E^2\right) \\
- \left(W2+W4+W5\right)\left[I\cdot W8 + I\cdot W9 + \left(I + IJ + \tau\right)W10\right] \\
- \left(W7+W9+W10\right)\left[I\cdot W3 + I\cdot W4 + \left(I + IJ + \tau\right)W5\right] \\
+ 2\left[I\cdot W9 + \left(I + IJ + \tau\right)W10\right]
\end{pmatrix}^2
$$

$$
\times \begin{pmatrix}
+ \left(W2+W4+W5\right)\left[I\cdot W3+I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right)^2 \\
- \left[I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right)^2 \\
- \left(W2+W4+W5\right)\left[I\cdot W8+I\cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right) \\
- \left(W7+W9+W10\right)\left[I\cdot W3+I\cdot W4+\left(I+IJ+\tau\right)W5\right]\left(\sigma_E^2\right) \\
+ 2\left[I\cdot W9+\left(I+IJ+\tau\right)W10\right]\left(\sigma_E^2\right) \\
+ \left(W7+W9+W10\right)\left[I\cdot W8+I\cdot W9+\left(I+IJ+\tau\right)W10\right]
\end{pmatrix}^{-2} .
$$

APPENDIX O

APPROXIMATION ANALYSIS PROGRAM

Table O.1 - Input / Output Device Designation

#	Use	Description
1	Input	Sample Sufficient Statistics
5	Input	*SOURCE*
6	Output	*SINK*
7	Output	Log of Screen Displays
8	Output	Bayesian Predictive Distribution
9	Output	Comparable Distributions

```
C*************************************************************
C  main program for Bayesian analysis of 2-way REM model,
C  posterior distribution of new row mean
C*************************************************************
C  variable definition
C    X(1) = row variance,    sigma(r)
C    X(2) = column variance, sigma(c)
C    X(3) = error variance,  sigma(e)
C*************************************************************
      PROGRAM BAYSRM
      REAL*8        I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      REAL*8        MEAN(2),STDDEV(2),PMF(201,2),NEW(201)
      COMMON /OUT/ MEAN,STDDEV,PMF,NEW
      REAL*8 CHUNK
      INTEGER TYPE
C
      CALL ERSET(3,0,-1)
C
C  get prior paramters and sample data
C
      CALL INPUTS
C
C  calculate moments for sampling theory predition distribution
C
      CALL SMPDAT
C
C  calculate posterior moments
C
      CALL MOMNTS(TYPE)
C
C  estimate posterior marginal dist'n of new row mean
C
      CALL ESTMAT(TYPE,CHUNK)
C
C  print selected percentiles of distributions
C
```

```
      CALL PRCNTL
C
C  write comparable distributions to file
C
      CALL COMPAR(TYPE,CHUNK)
C
      STOP
      END
C*************************************************************
C  subroutine to calculate posterior moments
C*************************************************************
      SUBROUTINE MOMNTS(TYPE)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      REAL*8       MEAN(2),STDDEV(2),PMF(201,2),NEW(201)
      COMMON /OUT/ MEAN,STDDEV,PMF,NEW
      INTEGER IP(7),TYPE
      REAL*8 CHUNK,DETERM,DETDEN,DETNUM,LFIDEN,LFINUM,RP(7),
     &       G(3),S(3),H(3,3),X(3),LB(3),UB(3)
      EXTERNAL DETERM,GRAD1,HESS1,LFNC1
C
C  calculate posterior mean
C
      MEAN(1) = (I*J*YDOTDT+TAU*MU)/(I*J+TAU)
C
C  calculate standard deviation
C
      W(10) = 0.D0
      S(1) = 1.D0
      S(2) = 1.D0
      S(3) = 1.D0
C
C  =>  for numerator
C
      W(5) = -1.D0
      G(3) = SSE/((I-1.D0)*(J-1.D0))
      G(1) = (SSR/(I-1.D0)-G(3))/J
      G(2) = (SSC/(J-1.D0)-G(3))/I
      IP(1) = 0
      CALL DBCOAH(LFNC1,GRAD1,HESS1,3,G,1,LB,UB,S,1.D0,IP,RP,
     &            X,LFINUM)
      WRITE(6,*)'num ',X
C
      IF (X(1).GT.0.D0) THEN
        IF (X(2).GT.0.D0) THEN
          TYPE = 1
          CALL HESS1(3,X,H,3)
          LFINUM = -LFINUM-0.5D0*DLOG(DETERM(H))
        ELSE
          TYPE = 2
          CALL ESTIM2(X(1),X(3),LFINUM)
        END IF
      ELSE
        IF (X(2).GT.0.D0) THEN
          TYPE = 3
          CALL ESTIM3(X(2),X(3),LFINUM)
        ELSE
```

```
              TYPE = 4
              CALL ESTIM4(X(3),LFINUM)
            END IF
          END IF
          WRITE(6,1) TYPE
        1 FORMAT(/,' type = ',I1,/)
      C
      C  =>  for denominator
      C
          W(5) = 0.D0
          G(1) = X(1)
          G(2) = X(2)
          G(3) = X(3)
          IP(1) = 0
          CALL DBCOAH(LFNC1,GRAD1,HESS1,3,G,1,LB,UB,S,1.D0,IP,RP,
         &            X,LFIDEN)
          WRITE(6,*)'den ',X
      C
          IF (X(1).GT.0.D0) THEN
            IF (X(2).GT.0.D0) THEN
              IF (TYPE.NE.1) THEN
                TYPE = 1
                WRITE(6,2) TYPE
        2       FORMAT(/,' at var. den., type = ',I1,/)
              END IF
              CALL HESS1(3,X,H,3)
              LFIDEN = -LFIDEN-0.5D0*DLOG(DETERM(H))
            ELSE
              IF (TYPE.NE.2) THEN
                TYPE = 2
                WRITE(6,2) TYPE
              END IF
              CALL ESTIM2(X(1),X(3),LFIDEN)
            END IF
          ELSE
            IF (X(2).GT.0.D0) THEN
              IF (TYPE.NE.3) THEN
                TYPE = 3
                WRITE(6,2) TYPE
              END IF
              CALL ESTIM3(X(2),X(3),LFIDEN)
            ELSE
              IF (TYPE.NE.4) THEN
                TYPE = 4
                WRITE(6,2) TYPE
              END IF
              CALL ESTIM4(X(3),LFIDEN)
            END IF
          END IF
      C
      C  calculate standard deviation using LaPlace estimation
      C
          STDDEV(1) = DEXP(0.5D0*(LFINUM-DLOG(I*J+TAU)-LFIDEN))
      C
          WRITE(6,77) MEAN(1),STDDEV(1)
       77 FORMAT(/' Bayes mean = ',F30.10,/'     st.dev = ',F30.10)
      C
          RETURN
```

```
      END
C*****************************************************************
C  subroutine for estimation of posterior marginal distribution
C*****************************************************************
      SUBROUTINE ESTMAT(TYPE,CHUNK)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      REAL*8       MEAN(2),STDDEV(2),PMF(201,2),NEW(201)
      COMMON /OUT/ MEAN,STDDEV,PMF,NEW
      INTEGER INDEX,K,L,IP(7),TYPE
      REAL*8 CHUNK,DETERM,LFIX,OUT,RP(7),G(3),S(3),
     &       H(3,3),X(3),LB(3),UB(3)
      EXTERNAL DETERM,GRAD1,HESS1,LFNC1
C
      W(5) = 0.5D0
      S(1) = 1.D0
      S(2) = 1.D0
      S(3) = 1.D0
C
C  estimate function at mean (y..)
C
      NEW(101) = MEAN(1)
      PMF(101,1) = 1.D0
      W(10) = 0.D0
      G(3) = SSE/((I-1.D0)*(J-1.D0))
      G(1) = (SSR/(I-1.D0)-G(3))/J
      G(2) = (SSC/(J-1.D0)-G(3))/I
      IP(1) = 0
      CALL DBCOAH(LFNC1,GRAD1,HESS1,3,G,1,LB,UB,S,1.D0,IP,RP,
     &             X,LFIX)
      WRITE(6,*)0,X
C
      IF (X(1).GT.0.D0) THEN
        IF (X(2).GT.0.D0) THEN
          IF (TYPE.NE.1) THEN
            TYPE = 1
            WRITE(6,1) 0,TYPE
    1       FORMAT(/,' at ',I3,', type = ',I1,/)
          END IF
          CALL HESS1(3,X,H,3)
          CHUNK = -LFIX-0.5D0*DLOG(DETERM(H))
        ELSE
          IF (TYPE.NE.2) THEN
            TYPE = 2
            WRITE(6,1) 0,TYPE
          END IF
          CALL ESTIM2(X(1),X(3),CHUNK)
        END IF
      ELSE
        IF (X(2).GT.0.D0) THEN
          IF (TYPE.NE.3) THEN
            TYPE = 3
            WRITE(6,1) 0,TYPE
          END IF
          CALL ESTIM3(X(2),X(3),CHUNK)
        ELSE
          IF (TYPE.NE.4) THEN
```

```
                TYPE = 4
                WRITE(6,1) 0,TYPE
              END IF
              CALL ESTIM4(X(3),CHUNK)
            END IF
          END IF
C
C  estimate function at 100 points up to 5 std dev around mean
C
          DO 100 INDEX = 1,100
C
            RINDEX = DFLOAT(INDEX)/20.D0
            DELT = STDDEV(1)*RINDEX
            W(10) = ((I*J+TAU)/2.D0)*DELT**2
            K = 101+INDEX
            NEW(K) = NEW(101)+DELT
C
            G(1) = X(1)
            G(2) = X(2)
            G(3) = X(3)
            IP(1) = 0
            CALL DBCOAH(LFNC1,GRAD1,HESS1,3,G,1,LB,UB,S,1.D0,IP,RP,
     &                   X,LFIX)
          WRITE(6,*)INDEX,X
C
            IF (X(1).GT.0.D0) THEN
              IF (X(2).GT.0.D0) THEN
                IF (TYPE.NE.1) THEN
                  TYPE = 1
                  WRITE(6,1) INDEX,TYPE
                END IF
                CALL HESS1(3,X,H,3)
                PMF(K,1) = DEXP(-LFIX-0.5D0*DLOG(DETERM(H))-CHUNK)
              ELSE
                IF (TYPE.NE.2) THEN
                  TYPE = 2
                  WRITE(6,1) INDEX,TYPE
                END IF
                CALL ESTIM2(X(1),X(3),OUT)
                PMF(K,1) = DEXP(OUT-CHUNK)
              END IF
            ELSE
              IF (X(2).GT.0.D0) THEN
                IF (TYPE.NE.3) THEN
                  TYPE = 3
                  WRITE(6,1) INDEX,TYPE
                END IF
                CALL ESTIM3(X(2),X(3),OUT)
                PMF(K,1) = DEXP(OUT-CHUNK)
              ELSE
                IF (TYPE.NE.4) THEN
                  TYPE = 4
                  WRITE(6,1) INDEX,TYPE
                END IF
                CALL ESTIM4(X(3),OUT)
                PMF(K,1) = DEXP(OUT-CHUNK)
              END IF
            END IF
```

```
C
C  => symmetric function has same value below mean/median
C
         L = 101-INDEX
         NEW(L) = NEW(101)-DELT
         PMF(L,1) = PMF(K,1)
  100 CONTINUE
C
C  normalize function to proper probability distribution
C
      CALL NRMLIZ(1)
      WRITE(8,8)(NEW(K),PMF(K,1),K=1,201)
    8 FORMAT(F15.6,'t',F9.6)
C
      WRITE(6,3) MEAN(1),STDDEV(1)
    3 FORMAT(//,13X,'              posterior mean = ',F12.4,
     &        /,13X,'posterior standard deviation = ',F12.4)
C
      RETURN
      END
C***************************************************************
C  subroutine for estimation of posterior marginal distribution
C  type 2:  column mode = 0; row and error modes positive
C***************************************************************
      SUBROUTINE ESTIM2(G1,G2,OUT)
C
      INTEGER IP(7)
      REAL*8 H(2,2),V(2),LFIX,OUT,RP(7),S(2),G(2),LB(2),UB(2),
     &       G1,G2
      EXTERNAL GRAD2,HESS2,LFNC2
C
      S(1) = 1.D0
      S(2) = 1.D0
      G(1) = G1
      G(2) = G2
      IP(1) = 0
C
      CALL DBCOAH(LFNC2,GRAD2,HESS2,2,G,1,LB,UB,S,1.D0,IP,RP,
     &            V,LFIX)
      CALL HESS2(2,V,H,2)
      OUT = -LFIX-0.5D0*DLOG(H(1,1)*H(2,2)-H(1,2)*H(2,1))
C
      RETURN
      END
C***************************************************************
C  subroutine for estimation of posterior marginal distribution
C  type 3:  row mode = 0; column and error modes positive
C***************************************************************
      SUBROUTINE ESTIM3(G1,G2,OUT)
C
      INTEGER IP(7)
      REAL*8 H(2,2),V(2),LFIX,OUT,RP(7),S(2),G(2),LB(2),UB(2),
     &       G1,G2
      EXTERNAL GRAD3,HESS3,LFNC3
C
      S(1) = 1.D0
      S(2) = 1.D0
      G(1) = G1
```

```
      G(2) = G2
      IP(1) = 0
C
      CALL DBCOAH(LFNC3,GRAD3,HESS3,2,G,1,LB,UB,S,1.D0,IP,RP,
     &            V,LFIX)
      CALL HESS3(2,V,H,2)
      OUT = -LFIX-0.5D0*DLOG(H(1,1)*H(2,2)-H(1,2)*H(2,1))
C
      RETURN
      END
C***************************************************************
C  subroutine for estimation of posterior marginal distribution
C  type 4:  row mode = column mode = 0;  error mode positive
C***************************************************************
      SUBROUTINE ESTIM4(G1,OUT)
C
      INTEGER IP(7)
     REAL*8 H(1,1),V(1),LFIX,OUT,RP(7),S(1),G(1),LB(1),UB(1),G1
      EXTERNAL GRAD4,HESS4,LFNC4
C
      G(1) = G1
      S(1) = 1.D0
      IP(1) = 0
C
      CALL DBCOAH(LFNC4,GRAD4,HESS4,1,G,1,LB,UB,S,1.D0,IP,RP,
     &            V,LFIX)
      CALL HESS4(1,V,H,1)
      OUT = -LFIX-0.5D0*DLOG(H(1,1))
C
      RETURN
      END
C***************************************************************
C  subroutine for calculation of distributions for comparison
C***************************************************************
      SUBROUTINE COMPAR(TYPE,CHUNK)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      REAL*8       MEAN(2),STDDEV(2),PMF(201,2),NEW(201)
      COMMON /OUT/ MEAN,STDDEV,PMF,NEW
      INTEGER K,IP(7),TYPE
      REAL*8 HIGH(2),LOW(2),MAX,MIN,STEP,H(3,3),X(3),
     &       CHUNK,DETERM,LFIX,OUT,RP(7),G(3),S(3),LB(3),UB(3)
      EXTERNAL DETERM,GRAD1,HESS1,LFNC1
C
C  find endpoints of interval
C
      LOW(1) = MEAN(1)-4.D0*STDDEV(1)
      HIGH(1) = MEAN(1)+4.D0*STDDEV(1)
      LOW(2) = MEAN(2)-4.D0*STDDEV(2)
      HIGH(2) = MEAN(2)+4.D0*STDDEV(2)
C
      IF (LOW(1).LT.LOW(2)) THEN
        MIN = LOW(1)
      ELSE
        MIN = LOW(2)
      ENDIF
      IF (HIGH(1).GT.HIGH(2)) THEN
```

```
            MAX = HIGH(1)
          ELSE
            MAX = HIGH(2)
          ENDIF
          STEP = (MAX-MIN)/201.D0
C
C  estimate distributions for each point in interval
C
          W(4) = 0.5D0
          W(5) = 0.5D0
          S(1) = 1.D0
          S(2) = 1.D0
          S(3) = 1.D0
          X(3) = SSE/((I-1.D0)*(J-1.D0))
          X(1) = (SSR/(I-1.D0)-X(3))/J
          X(2) = (SSC/(J-1.D0)-X(3))/I
C
          DO 100 K = 1,201
C
C  =>  find point in interval
C
            IF (K.EQ.1) THEN
              NEW(1) = MIN
            ELSE
              NEW(K) = NEW(K-1)+STEP
            END IF
C
C  =>  estimate posterior distribution
C
            IF (NEW(K).GE.LOW(1).AND.NEW(K).LE.HIGH(1)) THEN
              W(10) = ((I*J+TAU)/2.D0)*(NEW(K)-MEAN(1))**2
              G(1) = X(1)
              G(2) = X(2)
              G(3) = X(3)
              IP(1) = 0
              CALL DBCOAH(LFNC1,GRAD1,HESS1,3,G,1,LB,UB,S,1.D0,
     &                    IP,RP,X,LFIX)
C
              IF (X(1).GT.0.D0) THEN
                IF (X(2).GT.0.D0) THEN
                  IF (TYPE.NE.1) THEN
                    TYPE = 1
                    WRITE(6,1) K,TYPE
    1               FORMAT(/,' at ',I3,', type = ',I1,/)
                  END IF
                  CALL HESS1(3,X,H,3)
                 PMF(K,1) = DEXP(-LFIX-0.5D0*DLOG(DETERM(H))-CHUNK)
                ELSE
                  IF (TYPE.NE.2) THEN
                    TYPE = 2
                    WRITE(6,1) K,TYPE
                  END IF
                  CALL ESTIM2(X(1),X(3),OUT)
                  PMF(K,1) = DEXP(OUT-CHUNK)
                END IF
              ELSE
                IF (X(2).GT.0.D0) THEN
                  IF (TYPE.NE.3) THEN
```

```
                TYPE = 3
                WRITE(6,1) K,TYPE
              END IF
              CALL ESTIM3(X(2),X(3),OUT)
              PMF(K,1) = DEXP(OUT-CHUNK)
            ELSE
              IF (TYPE.NE.4) THEN
                TYPE = 4
                WRITE(6,1) K,TYPE
              END IF
              CALL ESTIM4(X(3),OUT)
              PMF(K,1) = DEXP(OUT-CHUNK)
            END IF
          END IF
        ELSE
          PMF(K,1) = 0.D0
        END IF
C
C  =>  estimate sampling theory distribution
C
        IF (NEW(K).GE.LOW(2).AND.NEW(K).LE.HIGH(2)) THEN
        PMF(K,2) = DEXP(-0.5D0*((NEW(K)-MEAN(2))/STDDEV(2))**2)
        ELSE
          PMF(K,2) = 0.D0
        END IF
  100 CONTINUE
C
C  normalize functions to proper probability distributions
C
      CALL NRMLIZ(1)
      CALL NRMLIZ(2)
C
      WRITE(6,902) MEAN,STDDEV
  902 FORMAT(//,' for comparable series:',
     &       //,' Bayes Posterior',5X,'Sampling Predictive',
     &       //,1X,F12.4,4X,'mean',4X,F12.4,
     &        /,1X,F12.4,2X,'std.dev.',2X,F12.4,//)
C
C  write series to file
C
      WRITE(9,4) (NEW(K),PMF(K,1),PMF(K,2),K=1,201)
    4 FORMAT(F15.6,'t',F9.6,'t',F9.6)
C
      RETURN
      END
C*****************************************************
C  subroutine for solicitation of prior parameters and
C  input of data
C*****************************************************
      SUBROUTINE INPUTS
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      REAL*8 ALPHA(3),BETA(3),GAMMA(3),SUM,SSQ,CSUMSQ,RSUMSQ
      CHARACTER*1 ANSWER,LCYES,UCYES
C
      LCYES = 'y'
      UCYES = 'Y'
```

```
C
C  solicit prior distribution parameters
C
  801 WRITE(6,802)
  802 FORMAT('1Use diffuse priors for all parameters? (y/n)')
      CALL FREAD(5,'S:',ANSWER,1)
      IF((ANSWER.EQ.UCYES).OR.(ANSWER.EQ.LCYES)) THEN
        MU = 0.D0
        TAU = 0.D0
        ALPHA(1) = 0.D0
        GAMMA(1) = 0.D0
        ALPHA(2) = 0.D0
        GAMMA(2) = 0.D0
        ALPHA(3) = 0.D0
        GAMMA(3) = 0.D0
      ELSE
        WRITE(6,811)
  811   FORMAT(//' Use diffuse prior for OVERALL MEAN? (y/n)')
        CALL FREAD(5,'S:',ANSWER,1)
        IF((ANSWER.EQ.UCYES).OR.(ANSWER.EQ.LCYES)) THEN
          MU = 0.D0
          TAU = 0.D0
        ELSE
  812     WRITE(6,813)
  813     FORMAT(//' Enter value for MU:')
          CALL FREAD(5,'R*8:',MU)
  815     WRITE(6,816)
  816     FORMAT(//' Enter value for TAU:')
          CALL FREAD(5,'R*8:',TAU)
          IF(TAU.LE.0.D0) THEN
            WRITE(6,817)
  817       FORMAT(' ERROR:  Value must exceed zero!!')
            GOTO 815
          END IF
        END IF
C
        WRITE(6,821)
  821   FORMAT(//' Use diffuse prior for ROW VARIANCE? (y/n)')
        CALL FREAD(5,'S:',ANSWER,1)
        IF((ANSWER.EQ.UCYES).OR.(ANSWER.EQ.LCYES)) THEN
          ALPHA(1) = 0.D0
          GAMMA(1) = 0.D0
        ELSE
  822     WRITE(6,823)
  823     FORMAT(//' Enter value for ALPHA:')
          CALL FREAD(5,'R*8:',ALPHA(1))
          IF(ALPHA(1).LE.0.D0) THEN
            WRITE(6,817)
            GOTO 822
          END IF
  825     WRITE(6,826)
  826     FORMAT(//' Enter value for BETA:')
          CALL FREAD(5,'R*8:',BETA(1))
          IF(BETA(1).LE.0.D0) THEN
            WRITE(6,817)
            GOTO 825
          END IF
          GAMMA(1) = 1.D0/BETA(1)
```

```
            END IF
C
            WRITE(6,831)
  831 FORMAT(//' Use diffuse prior for COLUMN VARIANCE? (y/n)')
            CALL FREAD(5,'S:',ANSWER,1)
            IF((ANSWER.EQ.UCYES).OR.(ANSWER.EQ.LCYES)) THEN
              ALPHA(2) = 0.D0
              GAMMA(2) = 0.D0
            ELSE
  832         WRITE(6,833)
  833         FORMAT(//' Enter value for ALPHA:')
              CALL FREAD(5,'R*8:',ALPHA(2))
              IF(ALPHA(2).LE.0.D0) THEN
                WRITE(6,817)
                GOTO 832
              END IF
  835         WRITE(6,836)
  836         FORMAT(//' Enter value for BETA:')
              CALL FREAD(5,'R*8:',BETA(2))
              IF(BETA(2).LE.0.D0) THEN
                WRITE(6,817)
                GOTO 835
              END IF
              GAMMA(2) = 1.D0/BETA(2)
            END IF
C
            WRITE(6,841)
  841  FORMAT(//' Use diffuse prior for ERROR VARIANCE? (y/n)')
            CALL FREAD(5,'S:',ANSWER,1)
            IF((ANSWER.EQ.UCYES).OR.(ANSWER.EQ.LCYES)) THEN
              ALPHA(3) = 0.D0
              GAMMA(3) = 0.D0
            ELSE
  842         WRITE(6,843)
  843         FORMAT(//' Enter value for ALPHA:')
              CALL FREAD(5,'R*8:',ALPHA(3))
              IF(ALPHA(3).LE.0.D0) THEN
                WRITE(6,817)
                GOTO 842
              END IF
  845         WRITE(6,846)
  846         FORMAT(//' Enter value for BETA:')
              CALL FREAD(5,'R*8:',BETA(3))
              IF(BETA(3).LE.0.D0) THEN
                WRITE(6,817)
                GOTO 845
              END IF
              GAMMA(3) = 1.D0/BETA(3)
            END IF
          END IF
C
          WRITE(6,851) MU,TAU,(ALPHA(I),GAMMA(I),I=1,3)
  851 FORMAT(//' Prior Distribution Parameters',
     &       //'    For OVERALL MEAN:     MU = ',F30.10,
     &        /'                         TAU = ',F30.10,
     &        /'    For ROW VARIANCE:  ALPHA = ',F30.10,
     &        /'                      1/BETA = ',F30.10,
     &        /' For COLUMN VARIANCE:  ALPHA = ',F30.10,
```

```
     &         /'                             1/BETA = ',F30.10,
     &         /'  For ERROR VARIANCE:  ALPHA = ',F30.10,
     &         /'                             1/BETA = ',F30.10)
      WRITE(6,852)
  852 FORMAT(//' Change values? (y/n)')
      CALL FREAD(5,'S:',ANSWER,1)
      IF((ANSWER.EQ.UCYES).OR.(ANSWER.EQ.LCYES)) GOTO 801
      WRITE(7,851) MU,TAU,(ALPHA(I),GAMMA(I),I=1,3)
C
C  enter sample data set descriptive statistics
C
  900 WRITE(6,901)
  901 FORMAT(///' Read sample statistics from file? (y/n)')
      CALL FREAD(5,'S:',ANSWER,1)
      IF((ANSWER.EQ.UCYES).OR.(ANSWER.EQ.LCYES)) THEN
        READ(1,902) I,J,SUM,SSQ,CSUMSQ,RSUMSQ
  902   FORMAT(F25.0)
        YDOTDT = SUM/(I*J)
        SSC = CSUMSQ/I-I*J*YDOTDT**2
        SSR = RSUMSQ/J-I*J*YDOTDT**2
        SSE = SSQ-RSUMSQ/J-CSUMSQ/I+I*J*YDOTDT**2
      ELSE
  903   WRITE(6,904)
  904   FORMAT(/' Enter NUMBER OF ROWS:')
        CALL FREAD(5,'R*8:',I)
        IF(I.LE.0.D0) THEN
          WRITE(6,817)
          GOTO 903
        END IF
C
  913   WRITE(6,914)
  914   FORMAT(' Enter NUMBER OF COLUMNS:')
        CALL FREAD(5,'R*8:',J)
        IF(J.LE.0.D0) THEN
          WRITE(6,817)
          GOTO 913
        END IF
C
  923   WRITE(6,924)
  924   FORMAT(' Enter SSR:')
        CALL FREAD(5,'R*8:',SSR)
        IF(SSR.LE.0.D0) THEN
          WRITE(6,817)
          GOTO 923
        END IF
C
  933   WRITE(6,934)
  934   FORMAT(' Enter SSC:')
        CALL FREAD(5,'R*8:',SSC)
        IF(SSC.LE.0.D0) THEN
          WRITE(6,817)
          GOTO 933
        END IF
C
  943   WRITE(6,944)
  944   FORMAT(' Enter SSE:')
        CALL FREAD(5,'R*8:',SSE)
        IF(SSE.LE.0.D0) THEN
```

```
          WRITE(6,817)
          GOTO 943
        END IF
C
        WRITE(6,954)
  954   FORMAT(' Enter MEAN:')
        CALL FREAD(5,'R*8:',YDOTDT)
      END IF
C
      WRITE(6,961) I,J,SSR,SSC,SSE,YDOTDT
  961 FORMAT(//' Data set statistics:'
     &       //'                    # of rows =   I = ',F30.10,
     &        /'                # of columns =   J = ',F30.10,
     &        /'    sum of squares, rows = SSR = ',F30.10,
     &        /' sum of squares, columns = SSC = ',F30.10,
     &        /'   sum of squares, error = SSE = ',F30.10,
     &        /'            overall mean = Y.. = ',F30.10)
      WRITE(6,852)
      CALL FREAD(5,'S:',ANSWER,1)
      IF((ANSWER.EQ.UCYES).OR.(ANSWER.EQ.LCYES)) GOTO 900
      WRITE(7,961) I,J,SSR,SSC,SSE,YDOTDT
C
      WRITE(6,962) SSR/(I-1.D0),SSC/(J-1.D0),
     &             SSE/((I-1.D0)*(J-1.D0))
      WRITE(7,962) SSR/(I-1.D0),SSC/(J-1.D0),
     &             SSE/((I-1.D0)*(J-1.D0))
  962 FORMAT(/' MSR = ',F30.10,
     &       /' MSC = ',F30.10,
     &       /' MSE = ',F30.10)
C
C  set common exponent values
C
      W(1) = (I*J-I-J+2.D0*ALPHA(3)+3.D0)/2.D0
      W(2) = (I+2.D0*ALPHA(1)+1.D0)/2.D0
      W(3) = (J+2.D0*ALPHA(2)+1.D0)/2.D0
      W(4) = 0.5D0
      W(6) = SSE/2.D0+GAMMA(3)
      W(7) = SSR/2.D0+GAMMA(1)
      W(8) = SSC/2.D0+GAMMA(2)
      W(9) = (I*J*TAU*(MU-YDOTDT)**2)/(2.D0*(I*J+TAU))
C
      RETURN
      END
C*****************************************************************
C  subroutine for sampling theory results
C*****************************************************************
      SUBROUTINE SMPDAT
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      REAL*8       MEAN(2),STDDEV(2),PMF(201,2),NEW(201)
      COMMON /OUT/ MEAN,STDDEV,PMF,NEW
      INTEGER INDEX,IPARAM(7),K,L
      REAL*8 DELT,DET,DETERM,LNFX,R(3,3),RINDEX,RPARAM(7),
     & SUM,X(3),XGUESS(3),XSCALE(3),VAR
C
      XGUESS(3) = SSE/((I-1.D0)*(J-1.D0))
      XGUESS(2) = (SSC/(J-1.D0)-XGUESS(3))/I
```

```
      XGUESS(1) = (SSR/(I-1.D0)-XGUESS(3))/J
      WRITE (6,3) XGUESS
      WRITE (7,3) XGUESS
    3 FORMAT(//,'    row variance = ',F20.4,
     &       /,' column variance = ',F20.4,
     &       /,'  error variance = ',F20.4)
      IF (XGUESS(1).LT.0.D0) THEN
        WRITE(6,901)
        WRITE(7,901)
  901   FORMAT(/,' ALERT   using ZERO for ROW var.')
        XGUESS(1) = 0.D0
      END IF
      IF (XGUESS(2).LT.0.D0) THEN
        WRITE(6,902)
        WRITE(7,902)
  902   FORMAT(/,' ALERT   using ZERO for COLUMN var.')
        XGUESS(2) = 0.D0
      END IF
C
      MEAN(2) = YDOTDT
      VAR = (XGUESS(3)+J*XGUESS(1)
     &     +I*(J+1.D0)*XGUESS(2))/(I*J)
      STDDEV(2) = DSQRT(VAR)
      WRITE(6,2) MEAN(2),STDDEV(2)
    2 FORMAT(//,' Sampling theory mean = ',F12.4,
     &       /,'   standard deviation = ',F12.4,/)
C
      RETURN
      END
C*****************************************************************
C  subroutine to find selected percentiles
C*****************************************************************
      SUBROUTINE PRCNTL
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      REAL*8       MEAN(2),STDDEV(2),PMF(201,2),NEW(201)
      COMMON /OUT/ MEAN,STDDEV,PMF,NEW
      INTEGER INDEX,K
      REAL*8 C(10),PRCL(10,2),DELT,SUM
C
      DATA C/.005,.025,.05,.125,.25,.75,.875,.95,.975,.995/
C
C  find selected percentiles for posterior distribution
C
      K = 1
      SUM = 0.D0
      DO 300 INDEX=1,201
        SUM = SUM+PMF(INDEX,1)
        IF ((K.LE.10).AND.(SUM.GE.C(K))) THEN
          PRCL(K,1) = NEW(INDEX)
          K = K+1
        END IF
  300 CONTINUE
C
C  sampling theory prediction intervals
C
      PRCL(1,2) = MEAN(2)-2.576*STDDEV(2)
```

```
      PRCL(2,2)  = MEAN(2)-1.96*STDDEV(2)
      PRCL(3,2)  = MEAN(2)-1.645*STDDEV(2)
      PRCL(4,2)  = MEAN(2)-1.15*STDDEV(2)
      PRCL(5,2)  = MEAN(2)-0.674*STDDEV(2)
      PRCL(6,2)  = MEAN(2)+0.674*STDDEV(2)
      PRCL(7,2)  = MEAN(2)+1.15*STDDEV(2)
      PRCL(8,2)  = MEAN(2)+1.645*STDDEV(2)
      PRCL(9,2)  = MEAN(2)+1.96*STDDEV(2)
      PRCL(10,2)  = MEAN(2)+2.576*STDDEV(2)
C
C  display selected intervals
C
      WRITE(7,902) MEAN,STDDEV,
     & PRCL(5,1),PRCL(6,1),PRCL(5,2),PRCL(6,2),
     & PRCL(4,1),PRCL(7,1),PRCL(4,2),PRCL(7,2),
     & PRCL(3,1),PRCL(8,1),PRCL(3,2),PRCL(8,2),
     & PRCL(2,1),PRCL(9,1),PRCL(2,2),PRCL(9,2),
     & PRCL(1,1),PRCL(10,1),PRCL(1,2),PRCL(10,2)
      WRITE(6,902) MEAN,STDDEV,
     & PRCL(5,1),PRCL(6,1),PRCL(5,2),PRCL(6,2),
     & PRCL(4,1),PRCL(7,1),PRCL(4,2),PRCL(7,2),
     & PRCL(3,1),PRCL(8,1),PRCL(3,2),PRCL(8,2),
     & PRCL(2,1),PRCL(9,1),PRCL(2,2),PRCL(9,2),
     & PRCL(1,1),PRCL(10,1),PRCL(1,2),PRCL(10,2)
  902 FORMAT(///,9X,'Bayes Posterior',
     &            15X,'Sampling Predictive',/
     & /,9X,F12.4,9X,'mean',9X,F12.4,
     & /,9X,F12.4,7X,'std.dev.',7X,F12.4,
     & ///,' Comparable Intervals:',/
     & /,6X,'Bayesian Theory',24X,'Sampling Theory',
     & /,5X,'HPD Credible Set',22X,'Prediction Interval',
     & /,1X,24('*'),15X,24('*'),
     & /,4X,'Lower',8X,'Upper',5X,'probability',
     &   5X,'Lower',8X,'Upper',
     & /,1X,11('-'),2X,11('-'),15X,11('-'),2X,11('-'),
     & /,1X,F11.4,2X,F11.4,6X,'50%',6X,F11.4,2X,F11.4,
     & /,1X,F11.4,2X,F11.4,6X,'75%',6X,F11.4,2X,F11.4,
     & /,1X,F11.4,2X,F11.4,6X,'90%',6X,F11.4,2X,F11.4,
     & /,1X,F11.4,2X,F11.4,6X,'95%',6X,F11.4,2X,F11.4,
     & /,1X,F11.4,2X,F11.4,6X,'99%',6X,F11.4,2X,F11.4,/////)
C
      RETURN
      END
C*****************************************************************
C  subroutine for normalizing function to proper distribution
C         and for calculating mean and variance
C*****************************************************************
      SUBROUTINE NRMLIZ(M)
C
      REAL*8        MEAN(2),STDDEV(2),PMF(201,2),NEW(201)
      COMMON /OUT/ MEAN,STDDEV,PMF,NEW
      INTEGER K,M
      REAL*8 SUM
C
      SUM = PMF(1,M)
      DO 200 K = 2,201
       SUM = SUM+PMF(K,M)
  200 CONTINUE
```

```
      MEAN(M) = 0.D0
      STDDEV(M) = 0.D0
      DO 210 K = 1,201
       PMF(K,M) = PMF(K,M)/SUM
       MEAN(M) = MEAN(M)+NEW(K)*PMF(K,M)
       STDDEV(M) = STDDEV(M)+NEW(K)**2*PMF(K,M)
  210 CONTINUE
      STDDEV(M) = DSQRT(STDDEV(M)-MEAN(M)**2)
C
      RETURN
      END
C***************************************************************
C  function to calculate determinant of 3x3 matrix
C***************************************************************
      REAL FUNCTION DETERM*8(M)
C
      REAL*8 M(3,3)
C
      DETERM = +M(1,1)*M(2,2)*M(3,3)+M(1,2)*M(2,3)*M(3,1)
     &         +M(1,3)*M(2,1)*M(3,2)-M(1,1)*M(2,3)*M(3,2)
     &         -M(1,2)*M(2,1)*M(3,3)-M(1,3)*M(2,2)*M(3,1)
C
      RETURN
      END
C***************************************************************
C  log of inverse of function to be integrated
C  type #1 : all modes positive
C***************************************************************
      SUBROUTINE LFNC1(N,X,FVAL)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER N
      REAL*8 FVAL,X(3),V1,V2,V3,V4,V5
C
      V1 = X(3)
      V2 = X(3)+J*X(1)
      V3 = X(3)+I*X(2)
      V4 = X(3)+J*X(1)+I*X(2)
      V5 = X(3)+J*X(1)+(I+I*J+TAU)*X(2)
C
      FVAL = W(1)*DLOG(V1)+W(2)*DLOG(V2)+W(3)*DLOG(V3)
     &       +W(4)*DLOG(V4)+W(5)*DLOG(V5)
     &       +W(6)/V1+W(7)/V2+W(8)/V3+W(9)/V4+W(10)/V5
C
      RETURN
      END
C***************************************************************
C  gradient vector of log of inverse of function
C  type #1 : all modes positive
C***************************************************************
      SUBROUTINE GRAD1(N,X,G)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER N
      REAL*8 G(3),X(3),Q1,V1,V2,V3,V4,V5
C
```

```
      Q1 = I+I*J+TAU
      V1 = X(3)
      V2 = X(3)+J*X(1)
      V3 = X(3)+I*X(2)
      V4 = X(3)+J*X(1)+I*X(2)
      V5 = X(3)+J*X(1)+(I+I*J+TAU)*X(2)
C
      G(1) = J*((W(2)-W(7)/V2)/V2+(W(4)-W(9)/V4)/V4
     &          +(W(5)-W(10)/V5)/V5)
      G(2) = I*((W(3)-W(8)/V3)/V3+(W(4)-W(9)/V4)/V4)
     &        +(W(5)-W(10)/V5)*Q1/V5
      G(3) =  (W(1)-W(6)/V1)/V1+(W(2)-W(7)/V2)/V2
     &        +(W(3)-W(8)/V3)/V3+(W(4)-W(9)/V4)/V4
     &        +(W(5)-W(10)/V5)/V5
C
      RETURN
      END
C*****************************************************************
C  Hessian matrix of log of inverse of function
C  type #1 : all modes positive
C*****************************************************************
      SUBROUTINE HESS1(N,X,H,LDH)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER LDH,N
      REAL*8 H(3,3),X(3),Q1,V1,V2,V3,V4,V5
C
      Q1 = I+I*J+TAU
      V1 = X(3)
      V2 = X(3)+J*X(1)
      V3 = X(3)+I*X(2)
      V4 = X(3)+J*X(1)+I*X(2)
      V5 = X(3)+J*X(1)+(I+I*J+TAU)*X(2)
C
      H(1,1) = -J**2*((W(2)-2.D0*W(7)/V2)/V2**2
     &                +(W(4)-2.D0*W(9)/V4)/V4**2
     &                +(W(5)-2.D0*W(10)/V5)/V5**2)
      H(1,2) = -J*((W(4)-2.D0*W(9)/V4)*I/V4**2
     &             +(W(5)-2.D0*W(10)/V5)*Q1/V5**2)
      H(1,3) = -J*((W(2)-2.D0*W(7)/V2)/V2**2
     &             +(W(4)-2.D0*W(9)/V4)/V4**2
     &             +(W(5)-2.D0*W(10)/V5)/V5**2)
      H(2,1) = H(1,2)
      H(2,2) = -I**2*((W(3)-2.D0*W(8)/V3)/V3**2
     &                +(W(4)-2.D0*W(9)/V4)/V4**2)
     &           -(W(5)-2.D0*W(10)/V5)*(Q1/V5)**2
      H(2,3) = -I*((W(3)-2.D0*W(8)/V3)/V3**2
     &             +(W(4)-2.D0*W(9)/V4)/V4**2)
     &           -(W(5)-2.D0*W(10)/V5)*Q1/V5**2
      H(3,1) = H(1,3)
      H(3,2) = H(2,3)
      H(3,3) = -(W(1)-2.D0*W(6)/V1)/V1**2
     &           -(W(2)-2.D0*W(7)/V2)/V2**2
     &           -(W(3)-2.D0*W(8)/V3)/V3**2
     &           -(W(4)-2.D0*W(9)/V4)/V4**2
     &           -(W(5)-2.D0*W(10)/V5)/V5**2
C
```

```
      RETURN
      END
C****************************************************************
C  log of inverse of function to be integrated
C  type #2 : column mode = 0 ; row and error modes positive
C****************************************************************
      SUBROUTINE LFNC2(N,X,FVAL)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER N
      REAL*8 FVAL,X(2),Q2,Q3,Q4,Q5,V1,V2,DEN2
C
      Q2 = I*W(4)+(I+I*J+TAU)*W(5)
      Q3 = I*W(9)+(I+I*J+TAU)*W(10)
      Q4 = W(2)+W(4)+W(5)
      Q5 = W(7)+W(9)+W(10)
      V1 = X(2)
      V2 = X(2)+J*X(1)
      DEN2 = (W(3)-W(8)/V1)*I/V1-(Q2-Q3/V2)/V2
C
      FVAL = (W(1)+W(3))*DLOG(V1)+Q4*DLOG(V2)+(W(6)+W(8))/V1
     &      +Q5/V2+DLOG(DEN2)
C
      RETURN
      END
C****************************************************************
C  gradient vector of log of inverse of function
C  type #2 : column mode = 0 ; row and error modes positive
C****************************************************************
      SUBROUTINE GRAD2(N,X,G)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER N
      REAL*8 G(2),X(2),Q2,Q3,Q4,Q5,V1,V2,DEN2
C
      Q2 = I*W(4)+(I+I*J+TAU)*W(5)
      Q3 = I*W(9)+(I+I*J+TAU)*W(10)
      Q4 = W(2)+W(4)+W(5)
      Q5 = W(7)+W(9)+W(10)
      V1 = X(2)
      V2 = X(2)+J*X(1)
      DEN2 = (W(3)-W(8)/V1)*I/V1-(Q2-Q3/V2)/V2
C
      G(1) = J*((Q4-Q5/V2)/V2-(Q2-2.D0*Q3/V2)/V2**2/DEN2)
      G(2) = (W(1)+W(3)-(W(6)+W(8))/V1)/V1+(Q4-Q5/V2)/V2
     &      -((W(3)-2.D0*W(8)/V1)*I/V1**2
     &             +(Q2-2.D0*Q3/V2)/V2**2)/DEN2
C
      RETURN
      END
C****************************************************************
C  Hessian matrix of log of inverse of function
C  type #2 : column mode = 0 ; row and error modes positive
C****************************************************************
      SUBROUTINE HESS2(N,X,H,LDH)
C
```

```
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER LDH,N
      REAL*8 H(2,2),X(2),Q2,Q3,Q4,Q5,V1,V2,DEN2
C
      Q2 = I*W(4)+(I+I*J+TAU)*W(5)
      Q3 = I*W(9)+(I+I*J+TAU)*W(10)
      Q4 = W(2)+W(4)+W(5)
      Q5 = W(7)+W(9)+W(10)
      V1 = X(2)
      V2 = X(2)+J*X(1)
      DEN2 = (W(3)-W(8)/V1)*I/V1-(Q2-Q3/V2)/V2
C
      H(1,1) = -(Q4-2.D0*Q5/V2)*(J/V2)**2
     &         +(Q2-3.D0*Q5/V2)*2.D0*J/V2**3/DEN2
     &         -((Q2-2.D0*Q3/V2)*J/V2**2/DEN2)**2
      H(1,2) = -(Q4-2.D0*Q5/V2)*J/V2**2
     &         +(Q2-3.D0*Q3/V2)*2.D0*J/V2**3/DEN2
     &         -(Q2-2.D0*Q3/V2)*J/V2**2
     &          *((W(3)-2.D0*W(8)/V1)*I/V1**2
     &                  +(Q2-2.D0*Q3/V2)/V2**2)/DEN2**2
      H(2,1) = H(1,2)
      H(2,2) = -(W(1)+W(3)-2.D0*(W(6)+W(8))/V1)/V1**2
     &         -(Q4-2.D0*Q5/V2)/V2**2
     &         +2.D0*((W(3)-3.D0*W(8)/V1)*I/V1**3
     &                     +(Q2-3.D0*Q3/V2)/V2**3)/DEN2
     &         -(((W(3)-2.D0*W(8)/V1)*I/V1**2
     &                  +(Q2-2.D0*Q3/V2)/V2**2)/DEN2)**2
C
      RETURN
      END
C*****************************************************************
C  log of inverse of function to be integrated
C  type #3 : row mode = 0 ; column and error modes positive
C*****************************************************************
      SUBROUTINE LFNC3(N,X,FVAL)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER N
      REAL*8 FVAL,X(2),V1,V3,V6,DEN3
C
      V1 = X(2)
      V3 = X(2)+I*X(1)
      V6 = X(2)+(I+I*J+TAU)*X(1)
      DEN3 = (W(2)-W(7)/V1)/V1-(W(4)-W(9)/V3)/V3
     &      -(W(5)-W(10)/V6)/V6
C
      FVAL = (W(1)+W(2))*DLOG(V1)+(W(3)+W(4))*DLOG(V3)
     &      +W(5)*DLOG(V6)+(W(6)+W(7))/V1+(W(8)+W(9))/V3
     &      +W(10)/V6+DLOG(J*DEN3)
C
      RETURN
      END
C*****************************************************************
C  gradient vector of log of inverse of function
C  type #3 : row mode = 0 ; column and error modes positive
C*****************************************************************
```

```
      SUBROUTINE GRAD3(N,X,G)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER N
      REAL*8 G(2),X(2),Q1,V1,V3,V6,DEN3
C
      Q1 = I+I*J+TAU
      V1 = X(2)
      V3 = X(2)+I*X(1)
      V6 = X(2)+(I+I*J+TAU)*X(1)
      DEN3 = (W(2)-W(7)/V1)/V1-(W(4)-W(9)/V3)/V3
     &       -(W(5)-W(10)/V6)/V6
C
      G(1) = (W(3)+W(4)-(W(8)+W(9))/V3)*I/V3
     &       +(W(5)-W(10)/V6)*Q1/V6-((W(4)-2.D0*W(9)/V3)*I/V3**2
     &       +(W(5)-2.D0*W(10)/V6)*Q1/V6**2)/DEN3
      G(2) = (W(1)+W(2)-(W(6)+W(7))/V1)/V1
     &       +(W(3)+W(4)-(W(8)+W(9))/V3)/V3+(W(5)-W(10)/V6)/V6
     &       -((W(2)-2.D0*W(7)/V1)/V1**2
     &       +(W(4)-2.D0*W(9)/V3)/V3**2
     &       +(W(5)-2.D0*W(10)/V6)/V6**2)/DEN3
C
      RETURN
      END
C****************************************************************
C  Hessian matrix of log of inverse of function
C  type #3 : row mode = 0 ; column and error modes positive
C****************************************************************
      SUBROUTINE HESS3(N,X,H,LDH)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER LDH,N
      REAL*8 H(2,2),X(2),Q1,V1,V3,V6,DEN3
C
      Q1 = I+I*J+TAU
      V1 = X(2)
      V3 = X(2)+I*X(1)
      V6 = X(2)+(I+I*J+TAU)*X(1)
      DEN3 = (W(2)-W(7)/V1)/V1-(W(4)-W(9)/V3)/V3
     &       -(W(5)-W(10)/V6)/V6
C
      H(1,1) = -(W(3)+W(4)-2.D0*(W(8)+W(9))/V3)*(I/V3)**2
     &           -(W(5)-2.D0*W(10)/V6)*(Q1/V6)**2
     &           +2.D0*((W(4)-3.D0*W(9)/V3)*I**2/V3**3
     &                 +(W(5)-3.D0*W(10)/V6)*Q1**2/V6**3)/DEN3
     &           -(((W(4)-2.D0*W(9)/V3)*I/V3**2
     &             +(W(5)-2.D0*W(10)/V6)*Q1/V6**2)/DEN3)**2
      H(1,2) = -(W(3)+W(4)-2.D0*(W(8)+W(9))/V3)*I/V3**2
     &           -(W(5)-2.D0*W(10)/V6)*Q1/V6**2
     &           +2.D0*((W(4)-3.D0*W(9)/V3)*I/V3**3
     &                 +(W(5)-3.D0*W(10)/V6)*Q1/V6**3)/DEN3
     &           -((W(2)-2.D0*W(7)/V1)/V1**2
     &            +(W(4)-2.D0*W(9)/V3)/V3**2
     &            +(W(5)-2.D0*W(10)/V6)/V6**2)
     &            *((W(4)-2.D0*W(9)/V3)*I/V3**2
     &             +(W(5)-2.D0*W(10)/V6)*Q1/V6**2)/DEN3**2
```

```
      H(2,1) = H(1,2)
      H(2,2) = -(W(1)+W(2)-2.D0*(W(6)+W(7))/V1)/V1**2
     &         -(W(3)+W(4)-2.D0*(W(8)+W(9))/V3)/V3**2
     &         -(W(5)-2.D0*W(10)/V6)/V6**2
     &         +2.D0*((W(2)-3.D0*W(7)/V1)/V1**3
     &               +(W(4)-3.D0*W(9)/V3)/V3**3
     &               +(W(5)-3.D0*W(10)/V6)/V6**3)/DEN3
     &         -(((W(2)-2.D0*W(7)/V1)/V1**2
     &           +(W(4)-2.D0*W(9)/V3)/V3**2
     &           +(W(5)-2.D0*W(10)/V6)/V6**2)/DEN3)**2
C
      RETURN
      END
C***************************************************************
C  log of inverse of function to be integrated
C  type #4 : row mode = column mode = 0 ; error mode positive
C***************************************************************
      SUBROUTINE LFNC4(N,V1,FVAL)
C
      REAL*8        I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/  I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER N
      REAL*8 DEN4,FVAL,Q2,Q3,Q4,Q5,Q6,Q7,V1
C
      Q2  = I*W(4)+(I+I*J+TAU)*W(5)
      Q3  = I*W(9)+(I+I*J+TAU)*W(10)
      Q4  = W(2)+W(4)+W(5)
      Q5  = W(7)+W(9)+W(10)
      Q6  = I*W(3)+I*W(4)+(I+I*J+TAU)*W(5)
      Q7  = I*W(8)+I*W(9)+(I+I*J+TAU)*W(10)
      DEN4 = (Q4*Q6-Q2)*V1**2-(Q4*Q7+Q5*Q6-2.D0*Q3)*V1+Q5*Q7
C
      FVAL = (W(1)+W(3)+Q4-4.D0)*DLOG(V1)+(W(6)+W(8)+Q5)/V1
     &      +DLOG(J*DEN4)
C
      RETURN
      END
C***************************************************************
C  gradient vector of log of inverse of function
C  type #4 : row mode = column mode = 0 ; error mode positive
C***************************************************************
      SUBROUTINE GRAD4(N,V1,G)
C
      REAL*8        I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/  I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER N
      REAL*8 DEN4,G(1),Q2,Q3,Q4,Q5,Q6,Q7,V1
C
      Q2  = I*W(4)+(I+I*J+TAU)*W(5)
      Q3  = I*W(9)+(I+I*J+TAU)*W(10)
      Q4  = W(2)+W(4)+W(5)
      Q5  = W(7)+W(9)+W(10)
      Q6  = I*W(3)+I*W(4)+(I+I*J+TAU)*W(5)
      Q7  = I*W(8)+I*W(9)+(I+I*J+TAU)*W(10)
      DEN4 = (Q4*Q6-Q2)*V1**2-(Q4*Q7+Q5*Q6-2.D0*Q3)*V1+Q5*Q7
C
      G(1) = (W(1)+W(3)+Q4-4.D0-(W(6)+W(8)+Q5)/V1)/V1
     &      +(2.D0*V1*(Q4*Q6-Q2)-Q4*Q7-Q5*Q6+2.D0*Q3)/DEN4
```

```
C
      RETURN
      END
C*************************************************************
C  Hessian matrix of log of inverse of function
C  type #4 : row mode = column mode = 0 ; error mode positive
C*************************************************************
      SUBROUTINE HESS4(N,V1,H,LDH)
C
      REAL*8       I,J,MU,SSC,SSE,SSR,TAU,W(10),YDOTDT
      COMMON /INN/ I,J,MU,SSC,SSE,SSR,TAU,W,YDOTDT
      INTEGER LDH,N
      REAL*8 H(1,1),Q2,Q3,Q4,Q5,Q6,Q7,V1,DEN4
C
      Q2 = I*W(4)+(I+I*J+TAU)*W(5)
      Q3 = I*W(9)+(I+I*J+TAU)*W(10)
      Q4 = W(2)+W(4)+W(5)
      Q5 = W(7)+W(9)+W(10)
      Q6 = I*W(3)+I*W(4)+(I+I*J+TAU)*W(5)
      Q7 = I*W(8)+I*W(9)+(I+I*J+TAU)*W(10)
      DEN4 = (Q4*Q6-Q2)*V1**2-(Q4*Q7+Q5*Q6-2.D0*Q3)*V1+Q5*Q7
C
      H(1,1) =-(W(1)+W(3)+Q4-4.D0-2.D0*(W(6)+W(8)+Q5)/V1)/V1**2
     &        +2.D0*(Q4*Q6-Q2)/DEN4
     &      -((2.D0*V1*(Q4*Q6-Q2)-Q4*Q7-Q5*Q6+2.D0*Q3)/DEN4)**2
C
      RETURN
      END
```

APPENDIX P

SUFFICIENT STATISTICS PROGRAM

Table P.1 - Input / Output Device Designation

#	Use	Description
1	Input	Earnings Matrix
2	Output	Sample Sufficient Statistics
5	Input	*SOURCE*
6	Output	*SINK*

```
C****************************************************
C  program to calculate sample sufficient statistics
C     enter data by ROW
C****************************************************
      INTEGER I,J,NCOL,NROW
      REAL*8 CSUM(50),RNCOL,RNROW,CSUMSQ,RSUMSQ,
     & RSUM(2000),SUM,SSQ,Y(2000,50)
C
      WRITE(6,901)
  901 FORMAT(' # of rows?')
      CALL FREAD(5,'I:',NROW)
      RNROW = DFLOAT(NROW)
      WRITE(6,902)
  902 FORMAT(' # of columns?')
      CALL FREAD(5,'I:',NCOL)
      RNCOL = DFLOAT(NCOL)
C
      READ(1,903) (Y(1,J),J=1,NCOL)
  903 FORMAT(50F3.0)
      CSUM(1)  = Y(1,1)
      RSUM(1)  = Y(1,1)
      SUM = Y(1,1)
      SSQ = Y(1,1)**2
      DO 100 J=2,NCOL
       CSUM(J)  = Y(1,J)
       RSUM(1)  = RSUM(1)+Y(1,J)
       SUM = SUM+Y(1,J)
       SSQ = SSQ+Y(1,J)**2
  100 CONTINUE
C
      DO 200 I=2,NROW
       READ(1,903) (Y(I,J),J=1,NCOL)
       CSUM(1)  = CSUM(1)+Y(I,1)
       RSUM(I)  = Y(I,1)
       SUM = SUM+Y(I,1)
       SSQ = SSQ+Y(I,1)**2
       DO 200 J=2,NCOL
```

```
              CSUM(J) = CSUM(J)+Y(I,J)
              RSUM(I) = RSUM(I)+Y(I,J)
              SUM = SUM+Y(I,J)
              SSQ = SSQ+Y(I,J)**2
  200 CONTINUE
C
      RSUMSQ = RSUM(1)**2
      DO 300 I=2,NROW
  300 RSUMSQ = RSUMSQ+RSUM(I)**2
C
      CSUMSQ = CSUM(1)**2
      DO 400 J=2,NCOL
  400 CSUMSQ = CSUMSQ+CSUM(J)**2
C
      WRITE(2,910) RNROW,RNCOL,SUM,SSQ,CSUMSQ,RSUMSQ
  910 FORMAT(F30.10)
C
      STOP
      END
```

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